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Composition of variable co-efficient singular integral operators

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Abstract. In this paper we give a formula for composition of two singular integral operators with variable co-efficients by explicitly calculating the lower order terms. Also we discuss the boundedness of the lower order terms in L^p-spaces.

Keywords. Singular integrals; Fourier transform L^p-spaces; convolutions.

1. Introduction

Singular integral operators, extensively studied by Calderon and Zygmund [1] are well known. They are principal value convolutions of the form

$$Tf(x) = af(x) + \lim_{\delta \to 0} \int_{|y| \geq \delta} f(x-y)K(y)\,dy \tag{1}$$

where the kernel K is homogeneous of degree $-n$, locally integrable away from the origin and has a mean value zero over the unit sphere. It is known for a long time that the composition of two singular integral operators of the above type is again a singular integral operator of the same type. But the relationship between the kernels is not clearly known.

In a recent paper Strichartz [3] takes a different view of the singular integrals. He considers the kernels being determined uniquely by their restrictions to the hyperplane $x_0 = 1$. Indeed, introducing the notation $|t|_\varepsilon^a = |t|^a (sgnt)^\varepsilon$ for any real t and a and $\varepsilon = 0$ or 1, a kernel $K(x_0, x)$ homogeneous of degree $-n-1$ and of parity ε is uniquely determined by $K(1, x)$ as the identity

$$K(x_0, x) = |x_0|_\varepsilon^{-n-1} K(1, x_0^{-1}x)$$

shows. So it is natural to consider kernels of the form

$$K(x_0, x) = |x_0|_\varepsilon^{-n-1} \phi(x_0^{-1}x)$$

for given ε and ϕ. He considers singular integrals of the form

$$S(\phi, \varepsilon)f(x_0, x) = \lim_{\delta \to 0} \int \int_{|y_0| \geq \delta} f(x_0 - y_0, x - y) K(\phi, \varepsilon)(y_0, y)\,dy_0\,dy \tag{2}$$

where

$$K(\phi, \varepsilon)(x_0, x) = |x_0|_\varepsilon^{-n-1} \phi(x_0^{-1}x).$$

This type of singular integrals and those of form (1) differs only by a term of the form af, a being a constant.

The conditions on $K(\phi, \varepsilon)$ are transformed into conditions on ϕ. The local integrability of $K(\phi, \varepsilon)$ away from 0 is equivalent to the integrability of ϕ and the mean value zero condition requires $\int \phi = 0$ when $\varepsilon = 0$ and is automatically satisfied when $\varepsilon = 1$. Given two integrable functions ϕ and ψ, and two parities ε_1 and ε_2, he considers the composition of the two singular integral operators $S(\phi, \varepsilon_1)\, S(\psi, \varepsilon_2)$ and proves that the composition is given by

$$S(\phi, \varepsilon_1)\, S(\psi, \varepsilon_2) f = c_0 f + S(A, \varepsilon_1 + \varepsilon_2) f$$

where c_0 is a constant and A is given in terms of ϕ and ψ by

$$A(x) = \lim_{\delta \to 0} \int_{\delta \leqslant |\lambda| \leqslant \delta^{-1}} \int \frac{|\lambda + 1|_{\varepsilon_1 + \varepsilon_2}^n}{|\lambda|_{\varepsilon_2}} \phi(x + \lambda(x - y)) \psi(y) \mathrm{d}\lambda \, \mathrm{d}y \tag{3}$$

In this paper, we try to obtain similar results for the composition of variable coefficient singular integral operators. As is well-known, the product of two such singular integrals is not a singular integral of the same form. So we expect that in the composition formula lower order terms will creep in. We consider singular integral operators whose Kernels are of the form

$$K(x_0, x, z_0, z) = b(x_0, x) |z_0|_\varepsilon^{-n-1} \phi(z_0^{-1} z), \tag{4}$$

where b is a given function and ϕ is integrable with $\int \phi = 0$ when $\varepsilon = 0$. The singular integral operator T with kernel $K(x_0, x, z_0, z)$ is then defined, as before, by

$$Tf(x_0, x) = \lim_{\delta \to 0} \int_{|z_0| \geqslant \delta} \int K(x_0, x, z_0, z) f(x_0 - z_0, x - z) \mathrm{d}z_0 \, \mathrm{d}z \tag{5}$$

Thus we repose our problem as follows: Given parities ε_1 and ε_2, integrable functions ϕ, ψ and functions a and b what will be the form of the composition $aS(\phi, \varepsilon_1) bS(\psi, \varepsilon_2)$? Without loss of generality, we assume that $a \equiv 1$ and the following is the main result of this paper.

THE MAIN THEOREM

Assume that $\phi, \psi \in D(R^n)$ satisfy $\int \phi = 0$ if $\varepsilon_1 = 0$ and $\int \psi = 0$ if $\varepsilon_2 = 0$ and that the function b in $C^{[(n+1)/2]+1}$ satisfy the following conditions:

(i) $D^{\alpha^*} b$ is bounded for any α^* with $|\alpha^*| \leqslant m = \left[\frac{n+1}{2}\right]$

(ii) $|D^{\alpha^*} b| \leqslant c(1 + |x_0|^2 + |x|^2)^{-1}$ for $|\alpha^*| = m + 1$ for all $(x_0, x) \in R^{n+1}$.

Then we have the following formula for any $f \in D(R^{n+1})$

$$\begin{aligned} &S(\phi, \varepsilon_1) bS(\psi, \varepsilon_2) f(x_0, x) \\ &\quad = a(x_0, x) f(x_0, x) + b(x_0, x) S(A, \varepsilon_1 + \varepsilon_2) f(x_0, x) \\ &\quad + \sum_{1 \leqslant |\alpha^*| \leqslant m} g_{\alpha^*}(x_0, x) T_{\alpha^*} f(x_0, x) + \sum_{|\alpha^*| = m+1} \frac{m+1}{\alpha^*!} I_{\alpha^*} f(x_0, x) \end{aligned} \tag{6}$$

where

$$a(x_0, x) = \left(-4 \int_2^\infty t^{-1} \log(t + 1/t - 1) \mathrm{d}t \right) (\int \phi)(\int \psi) b(x_0, x)$$

$$-4 \int_0^2 t^{-1} \log(1+t) \mathrm{d}t (\int \phi)(\int \psi) \tag{7}$$

$$A(x) = \lim_{\delta \to 0} \int \int_{\delta \leqslant |\lambda| \leqslant \delta^{-1}} \frac{|\lambda+1|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda|_{\varepsilon_2}} \phi(x + \lambda(x-y)) \psi(y) \mathrm{d}\lambda \, \mathrm{d}y$$

$$g_{\alpha^*}(x_0, x) = \frac{(-1)^{|\alpha^*|}}{\alpha^*!} (D^{\alpha^*} b)(x_0, x) \tag{8}$$

$$T_{\alpha^*} f(x_0, x) = \iint |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} A_{\alpha^*}(z_0^{-1} z) f(x_0 - z_0, x - z) \mathrm{d}z_0 \mathrm{d}z \tag{9}$$

with

$$A_{\alpha^*}(z) = \lim_{\delta \to 0} \int \int_{\delta \leqslant |\lambda| \leqslant \delta^{-1}} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k} (\lambda-1)^k y^\alpha \phi(y) \psi(z + (\lambda-1) \times (z-y)) \mathrm{d}\lambda \, \mathrm{d}y$$

(where we have put $\alpha^* = (\alpha_0, \alpha)$) and

$$I_{\alpha^*} f(x_0, x) = \int_0^1 (1-t)^m G_{\alpha^*} f(x_0, x) \mathrm{d}t \tag{10}$$

with

$$G_{\alpha^*} f(x_0, x) = \iint K_{\alpha^*}(x_0, x, z_0, z) f(x_0 - z_0, x - z) \mathrm{d}z_0 \, \mathrm{d}z$$

and $K_{\alpha^*}(x_0, x, z_0, z)$ is given by

$$K_{\alpha^*}(x_0, x, z_0, z) = \lim_{\delta \to 0} \int \int_{\delta \leqslant |y_0 - z_0| \leqslant \delta^{-1}} |y_0 - z_0|^{-n-1}_{\varepsilon_2} |y_0|^{-n-1}_{\varepsilon_1} y_0^{\alpha 0} y^\alpha$$

$$\phi(y_0^{-1} y)(D)^{\alpha^*} b)(x_0 - t y_0, x - t y) \psi(z - y/z_0 - y_0) \mathrm{d}y_0 \, \mathrm{d}y. \tag{11}$$

Further T_{α^*} is bounded from L^p into L^q where $1 < p < q < \infty, \frac{1}{q} = \frac{1}{p} - \frac{|\alpha^*|}{n+1}$ and I_{α^*} is bounded from L^p into L^q where $1 < p < q < \infty, \frac{1}{q} = \frac{1}{p} - \frac{m+1}{n+1}$.

2. The composition formula

For a given integrable function ϕ on R^n and a parity ε the kernel $K(\phi, \varepsilon)$ is defined on R^{n+1} by

$$K(\phi, \varepsilon)(x_0, x) = |x_0|^{-n-1}_\varepsilon \phi(x_0^{-1} x).$$

Then this function $K(\phi, \varepsilon)$ is homogeneous of degree $-n-1$ and equals ϕ on the

hyperplane $x_0 = 1$. When the parity ε is even, we assume $\int \phi(x)\,dx = 0$. The singular integral $S(\phi, \varepsilon)$ corresponding to the kernel $K(\phi, \varepsilon)$ is defined by

$$S(\phi, \varepsilon) f(x_0, x) = \lim_{\delta \to 0} \int \int_{|y_0| \geq \delta} K(\phi, \varepsilon)(y_0, y) f(x_0 - y_0, x - y)\,dy_0\,dy \quad (12)$$

for functions $f \in D(R^{n+1})$.

To find exactly how the composition $S(\phi, \varepsilon) b S(\psi, \varepsilon_2)$ will look like, we first assume that ϕ and ψ are in $D(R^n)$. Assume that ϕ, ψ and b satisfy all the conditions of the theorem. Then we have

LEMMA 2.1.

Define

$$K_\delta(x_0, x, z_0, z) = \int \int_{\substack{|y_0| \geq \delta \\ |y_0 - z_0| \geq \delta}} |y_0|_\varepsilon^{-n-1} |z_0 - y_0|_\varepsilon^{-n-1} \phi(y_0^{-1} y)$$
$$b(x_0 - y_0, x - y) \psi(z - y/z_0 - y_0)\,dy_0\,dy \quad (13)$$

Then for any $f \in D(R^{n+1})$ we have

$$S(\phi, \varepsilon_1) b S(\psi, \varepsilon_2) f = \lim_{\delta \to 0} K_\delta f \quad (14)$$

both pointwise and in the L^2 norm.

The proof of Lemma 2.1 is routine and we omit it here. (See [3] Lemma 2.1).

Now we write $K_\delta = A_\delta + B_\delta$ where

$$A_\delta(x_0, x, z_0, z) = K_\delta(x_0, x, z_0, z), \text{ if } |z_0| \geq 2\delta \quad (15)$$
$$= 0, \quad \text{if} \quad |z_0| < 2\delta$$

and

$$B_\delta(x_0, x, z_0, z) = K_\delta(x_0, x, z_0, z), \quad \text{if} \quad |z_0| < 2\delta$$
$$= 0, \quad \text{if} \quad |z_0| \geq 2\delta.$$

We will consider the convolution with each term separately. For the convolution $B_\delta * f$ we have:

LEMMA 2.2.

$$\lim_{\delta \to 0} B_\delta * f = a(\int \phi)(\int \psi) b f$$

in the L^2 norm where the constant a is given by

$$a = -4 \int_0^2 t^{-1} \log(1 + t)\,dt.$$

Proof. For $|z_0| < 2\delta$ we have after a change of variables

$$B_\delta(x_0, x, z_0, z) = \delta^{-n-1} \int \int_{\substack{|y_0| \geq 1 \\ |\delta^{-1}z_0 - y_0| \geq 1}} |y_0|_{\varepsilon_1}^{-n-1} |\delta^{-1}z_0 - y_0|_{\varepsilon_2}^{-n-1}$$
$$b(x_0 - \delta y_0, x - \delta y)\, \phi(y/y_0)\psi\left(\frac{\delta^{-1}z - y}{\delta^{-1}z_0 - y_0}\right) dy_0\, dy.$$

Therefore,

$$|B_\delta(x_0, x, z_0, z)| \leq c\delta^{-n-1} L(\delta^{-1}z_0, \delta^{-1}z)$$

where the function L is given by

$$L(z_0, z) = \int \int_{\substack{|y_0| \geq 1 \\ |z_0 - y_0| \geq 1}} |y_0|^{-n-1} |z_0 - y_0|^{-n-1} |\phi(y/y_0)|$$
$$\psi(z - y/z_0 - y_0)\, dy_0\, dy \qquad \text{if} \quad |z_0| < 2\delta$$
$$= 0, \qquad \text{if} \quad |z_0| \geq 2\delta.$$

It then follows that L is an integrable function and hence

$$\lim_{\delta \to 0} \iint B_\delta(x_0, x, z_0, z) f(x_0 - z_0, x - z)\, dz_0\, dz$$
$$= \left[\lim_{\delta \to 0} \iint B_\delta(x_0, x, z_0, z)\, dz_0\, dz\right] f(x_0, x).$$

Thus it remains to show

$$\lim_{\delta \to 0} \iint B_\delta(x_0, x, z_0, z)\, dz_0\, dz = a(\int \phi)(\int \psi)\, b(x_0, x).$$

Now

$$\iint B_\delta(x_0, x, z_0, z)\, dz_0 dz$$
$$= \int \int_{-2\delta}^{2\delta} dz_0 dz \int \int_{\substack{|y_0| \geq \delta \\ |z_0 - y_0| \geq \delta}} |y_0|_{\varepsilon_1}^{-n-1} |z_0 - y_0|_{\varepsilon_2}^{-n-1} b(x_0 - y_0, x - y)$$
$$\phi(y/y_0)\psi(z - y/z_0 - y_0)\, dy_0 dy$$
$$= \int \int_{-2}^{2} dz_0 dz \int_{\substack{|y_0| \geq 1 \\ |z_0 - y_0| \geq 1}} |y_0|_{\varepsilon_1}^{-n-1} |z_0 - y_0|_{\varepsilon_2}^{-n-1} b(x_0 - \delta y_0, x - \delta y)$$
$$\phi(y/y_0)\psi\left(\frac{z - y}{z_0 - y_0}\right) dy_0 dy.$$

Since

$$\int \int_{\substack{|y_0|\geq 1 \\ |z_0-y_0|\geq 1}} |y_0|_{\varepsilon_1}^{-n-1} |z_0-y_0|_{\varepsilon_2}^{-n-1} b(x_0-\delta y_0, x-\delta y)\phi(y/y_0)$$

$$\psi\left(\frac{z-y}{z_0-y_0}\right) dy_0\, dy$$

converges to

$$\int \int_{\substack{|y_0|\geq 1 \\ |z_0-y_0|\geq 1}} |y_0|_{\varepsilon_1}^{-n-1} |z_0-y_0|_{\varepsilon_2}^{-n-1} b(x_0, x)\phi(y/y_0)\psi\left(\frac{z-y}{z_0-y_0}\right) dy_0\, dy$$

as $\delta \to 0$ and is bounded by an integrable function, dominated convergence theorem shows that

$$\lim_{\delta\to 0} \iint B_\delta(x_0, x, z_0, z)\, dz_0 dz$$

$$= b(x_0, x) \int \int_{-2}^{2} dz_0 dz \int \int_{\substack{|y_0|\geq 1 \\ |z_0-y_0|\geq 1}} |y_0|_{\varepsilon_1}^{-n-1} |z_0-y_0|_{\varepsilon_2}^{-n-1}$$

$$\phi(y/y_0)\psi\left(\frac{z-y}{z_0-y_0}\right) dy_0 dy$$

$$= b(x_0, x)(\int \phi)(\int \psi) \int_{-2}^{2} \int_{\substack{|y_0|\geq 1 \\ |z_0-y_0|\geq 1}} |y_0|_{\varepsilon_1}^{-1} |z_0-y_0|_{\varepsilon_2}^{-1} dy_0\, dz_0.$$

We have to consider only the case $\varepsilon_1 = \varepsilon_2 = 1$ and in that case the change of variable $y_0 \to (y_0 + \frac{1}{2})z_0$ gives

$$\int_{-2}^{2} \int_{\substack{|y_0|\geq 1 \\ |z_0-y_0|\geq 1}} y_0^{-1}(z_0-y_0)^{-1}\, dy_0 dz_0$$

$$= 4 \int_0^2 \int_{\frac{1}{2}+\frac{1}{z_0}}^{\infty} (\tfrac{1}{2}+y_0)^{-1}(\tfrac{1}{2}-y_0)^{-1} z_0^{-1}\, dy_0\, dz_0$$

$$= -4 \int_0^2 z_0^{-1} \log(1+z_0)\, dz_0.$$

Q.E.D.

Turning our attention to the next term, we have for $|z_0| \geqslant 2\delta$

$$A_\delta(x_0, x, z_0, z) = \int \int\limits_{\substack{|y_0| \geqslant \delta \\ |z_0 - y_0| \geqslant \delta}} |y_0|_{\varepsilon_1}^{-n-1} |z_0 - y_0|_{\varepsilon_2}^{-n-1} b(x_0 - y_0, x - y)\phi(y/y_0)\psi\left(\frac{z-y}{z_0-y_0}\right) dy_0\, dy$$

Expanding $b(x_0 - y_0, x - y)$ in Taylor series about (x_0, x)

$$\begin{aligned} b(x_0 - y_0, x - y) &= b(x_0, x) + \sum_{1 \leqslant |\alpha^*| \leqslant m} \frac{(-1)^{|\alpha^*|}}{\alpha^*!} y_0^{\alpha_0} y^\alpha (D^{\alpha^*} b)(x_0, x) \\ &\quad + \sum_{|\alpha^*| = m+1} \frac{m+1}{\alpha^*!} \int_0^1 (1-t)^m y_0^{\alpha_0} y^\alpha (D^{\alpha^*} b)(x_0 - ty_0, x - ty)\, dt, \end{aligned} \tag{16}$$

where $\alpha^* = (\alpha_0, \alpha)$ is a multi index and $m = [\frac{1}{2}(n+1)]$. Thus we have for $|z_0| \geqslant 2\delta$

$$\begin{aligned} A_\delta(x_0, x, z_0, z) &= b(x_0, x) A_{0,\delta}(x_0, x, z_0, z) + \sum_{1 \leqslant |\alpha^*| \leqslant m} \frac{(-1)^{|\alpha^*|}}{\alpha^*!} (D^{\alpha^*} b)(x_0, x) \\ &\quad A_{\alpha^*,\delta}(x_0, x, z_0, z) + \sum_{|\alpha^*| = m+1} \frac{m+1}{\alpha^*!} \int_0^1 (1-t)^m A_{\alpha^*,\delta,t}(x_0, x, z_0, z)\, dt \end{aligned} \tag{17}$$

where we have set

$$A_{0,\delta}(x_0, x, z_0, z) = \int \int\limits_{\substack{|y_0| \geqslant \delta \\ |z_0 - y_0| \geqslant \delta}} |y_0|_{\varepsilon_1}^{-n-1} |z_0 - y_0|_{\varepsilon_2}^{-n-1} \phi(y/y_0)\psi\left(\frac{z-y}{z_0-y_0}\right) dy_0\, dy \tag{18}$$

$$A_{\alpha^*,\delta}(x_0, x, z_0, z) = \int \int\limits_{\substack{|y_0| \geqslant \delta \\ |z_0 - y_0| \geqslant \delta}} |y_0|_{\varepsilon_1}^{-n-1} |z_0 - y_0|_{\varepsilon_2}^{-n-1} \phi(y/y_0)\psi\left(\frac{z-y}{z_0-y_0}\right) dy_0\, dy, \tag{19}$$

$$A_{\alpha^*,\delta,t}(x_0, x, z_0, z) = \int \int\limits_{\substack{|y_0| \geqslant \delta \\ |z_0 - y_0| \geqslant \delta}} |y_0|_{\varepsilon_1}^{-n-1} |z_0 - y_0|_{\varepsilon_2}^{-n-1} y_0^{\alpha_0} y^\alpha (D^{\alpha^*} b)(x_0 - ty_0, x - ty)\phi(y/y_0)\psi\left(\frac{z-y}{z_0-y_0}\right) dy_0\, dy. \tag{20}$$

Then the following Lemma 2.3 follows immediately from Lemma 2.3 of [3].

LEMMA 2.3.

$$\lim_{\delta \to 0} A_{0,\delta^*} f = S(A, \varepsilon_1 + \varepsilon_2) f + a'(\int \phi)(\int \psi) f$$

both pointwise and in L^2 norm where

$$a' = -4 \int_2^\infty t^{-1} \log\left(\frac{t+1}{t-1}\right) dt$$

and

$$A(x) = \lim_{\delta \to 0} \int \int_{\delta \leqslant |\lambda| \leqslant \delta^{-1}} \frac{|\lambda+1|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda|_{\varepsilon_2}} \psi(y)\phi(x+\lambda(x-y)) d\lambda\, dy$$

the limit existing both pointwise and in L^1 norm.

Next we consider the terms $A_{\alpha^*,\delta^*} f$ for $1 \leqslant |\alpha^*| \leqslant m$. Setting $A_{\alpha^*,\delta} = A_{k,\delta}$ for convenience when $|\alpha^*| = k$, $1 \leqslant k \leqslant m$ we have the following proposition concerning the convolution

$$\lim_{\delta \to 0} A_{k,\delta} * f.$$

PROPOSITION 2.1.

$$\lim_{\delta \to 0} \iint A_{k,\delta}(x_0, x, z_0, z) f(x_0 - z_0, x - z)\, dz_0\, dz$$
$$= \iint |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} A_k(z_0^{-1} z) f(x_0 - z_0, x - z)\, dz_0\, dz,$$

where the function $A_k(z)$ is given by

$$A_k(z) = \lim_{\delta \to 0} \int \int_{\delta \leqslant |\lambda| \leqslant \delta^{-1}} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda - 1|_{\varepsilon_1}} \lambda^{-k} (\lambda - 1)^k y^\alpha \phi(y)$$
$$\psi(z + (\lambda - 1)(z - y))\, d\lambda dy \tag{21}$$

where the limit exists in the L^2 norm. Furthermore, the operator

$$f \to \iint |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} A_k(z_0^{-1} z) f(x_0 - z_0, x - z)\, dz_0\, dz$$

is bounded from L^p into L^q for $1 < p < q < \infty, \dfrac{1}{q} = \dfrac{1}{p} - \dfrac{k}{n+1}$

Proof. The proof follows in several steps.

Steps. A change of variables in (19) shows that

$$A_{k,\delta}(z_0, z) = |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} A_{k,\delta|z_0|^{-1}}(1, z_0^{-1} z) \tag{22}$$

where

$$A_{k,\delta}(1,z) = \int \int_{\substack{|y_0| \geqslant \delta \\ |1-y_0| \geqslant \delta}} |y_0|_{\varepsilon_1}^{-n-1} |1-y_0|_{\varepsilon_2}^{-n-1} y_0^{\alpha_0} y^{\alpha} \phi(y/y_0) \times \psi\left(\frac{z-y}{1-y_0}\right) \mathrm{d}y_0 \, \mathrm{d}y. \tag{23}$$

By changing y into $y_0 y$ and then putting $\lambda = y_0/(1-y_0)$ we obtain

$$A_{k,\delta}(1, z) = \int\int_{T_\delta} \frac{|\lambda+1|_{\varepsilon_1+\varepsilon_2}^{n}}{|\lambda|_{\varepsilon_1}} \lambda^k (\lambda+1)^{-k} y^{\alpha} \phi(y) \psi(z+\lambda(z-y)) \, \mathrm{d}\lambda \, \mathrm{d}y,$$

where T_δ is the union of the intervals

$$\left(\frac{\delta}{1-\delta}, \frac{1-\delta}{\delta}\right) \text{ and } \left(-\frac{1+\delta}{\delta}, -\frac{\delta}{1+\delta}\right).$$

Changing $\lambda+1$ into λ we obtain

$$A_{k,\delta}(1, z) = \int\int_{T_\delta} \frac{|\lambda|_{\varepsilon_1+\varepsilon_2}^{n}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k} (\lambda-1)^k y^{\alpha} \phi(y) \psi(z+(\lambda-1)(z-y)) \, \mathrm{d}x \, \mathrm{d}y$$

with the region T_δ given by

$$T_\delta = \left(\frac{1}{1-\delta}, \frac{1}{\delta}\right) \cup \left(-\frac{1}{\delta}, \frac{1}{1+\delta}\right).$$

Considering only $\delta < \frac{1}{2}$ we take $T_\delta = R_\delta \cup S_\delta$ where

$$R_\delta = \left(-2, \frac{1}{1+\delta}\right) \cup \left(\frac{1}{1-\delta}, 2\right)$$

and

$$S_\delta = \left(-\frac{1}{\delta}, -2\right) \cup \left(2, \frac{1}{\delta}\right).$$

We then write

$$\begin{aligned} A_{k,\delta}(1, z) &= \int\int_{R_\delta} \frac{|\lambda|_{\varepsilon_1+\varepsilon_2}^{n}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k} (\lambda-1)^k y^{\alpha} \phi(y) \psi(z+(\lambda-1)(z-y)) \mathrm{d}\lambda \mathrm{d}y \\ &+ \int\int_{S_\delta} \frac{|\lambda|_{\varepsilon_1+\varepsilon_2}^{n}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k} (\lambda-1)^k y^{\alpha} \phi(y) \psi(z+(\lambda-1)(z-y)) \, \mathrm{d}x \, \mathrm{d}y \qquad (24) \\ &= B_{k,\delta}(1, z) + C_{k,\delta}(1, z). \end{aligned}$$

Adding and subtracting appropriate terms we obtain

$$\begin{aligned} B_{k,\delta}(1, z) = \int\int_{R_\delta} \frac{|\lambda|_{\varepsilon_1+\varepsilon_2}^{n}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k} (\lambda-1)^k y^{\alpha} \phi(y) \{\psi(z+(\lambda-1)(z-y)) \\ - \psi(z)\} \, \mathrm{d}\lambda \mathrm{d}y + h_{k,\delta}(\textstyle\int y^{\alpha} \phi) \psi(z), \end{aligned} \tag{25}$$

where

$$h_{k,\delta} = \int_{R_\delta} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k \, d\lambda.$$

Likewise we obtain

$$C_{k,\delta}(1,z) = \int\int_{S_\delta} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k \psi(z+(\lambda-1)(z-y))[y^\alpha \phi(y) - z^\alpha \phi(z)] \, d\lambda \, dy + r_{k,\delta}(\int \psi) z^\alpha \phi(z), \tag{26}$$

where

$$r_{k,\delta} = \int_{S_\delta} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k |\lambda-1|^{-n} \, d\lambda. \tag{27}$$

Define a function $A_k(z)$ by

$$A_k(z) = \int\int_{|\lambda|\leqslant 2} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k y^\alpha \phi(y)[\psi(z+(\lambda-1)(z-y)) - \psi(z)] \, d\lambda \, dy + \int\int_{|\lambda|\geqslant 2} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k \psi(z+(\lambda-1)(z-y))$$
$$[y^\alpha \phi(y) - z^\alpha \phi(z)] \, d\lambda \, dy + h_k(\int y^\alpha \phi)\psi(z) + r_k(\int \psi) z^\alpha \phi(z) \tag{28}$$

with

$$h_k = \int_{|\lambda|\leqslant 2} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k \, d\lambda$$

and

$$r_k = \int_{|\lambda|\geqslant 2} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k \, d\lambda.$$

Step 2. Assume that both ϕ and ψ are supported in the ball $\{x: |x| \leqslant R\}$. Then

$$|A_k(z) - A_{k,\delta}(1,z)| \leqslant c_k \delta \quad \text{for} \quad |z| \leqslant 3R$$
$$A_k(z) = A_{k,\delta}(1,z) \quad \text{for} \quad |z| > 3R.$$

Writing

$$A_k(z) = A_{1,k}(z) + A_{2,k}(z) + h_k(\int y^\alpha \phi)\psi(z) + r_k(\int \psi) z^\alpha \phi(z). \tag{29}$$

We have

$$A_k(z) - A_{k,\delta}(1,z) = (A_{1,k}(z) - B_{k,\delta}(1,z)) + (A_{2,k}(z) - c_{k,\delta}(1,z)) + (h_k - h_{k,\delta})(\int y^\alpha \phi)\psi(z) + (r_k - r_{k\delta})(\int \psi) z^\alpha \phi(z), \tag{30}$$

where

$$A_{1,k}(z)-B_{k,\delta}(1,z)=\int_{1/(1+\delta)}^{1/(1-\delta)}\int\frac{|\lambda|^{n}_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}}\lambda^{-k}(\lambda-1)^{k}y^{\alpha}\phi(y)$$

$$[\psi(z+(\lambda-1)(z-y))-\psi(z)]\,\mathrm{d}\lambda\,\mathrm{d}y, \tag{31}$$

$$A_{2,k}(z)-c_{k,\delta}(1,z)=\int_{|\lambda|\geqslant\frac{1}{\delta}}\int\frac{|\lambda|^{n}_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}}\lambda^{-k}(\lambda-1)^{k}(y^{\alpha}\phi(y)$$

$$-z^{\alpha}\phi(z))\psi(z+(\lambda-1)(z-y))\,\mathrm{d}\lambda\,\mathrm{d}y, \tag{32}$$

$$h_k-h_{k\delta}=\int_{1/(1+\delta)}^{1/(1-\delta)}\frac{|\lambda|^{n}_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}}\lambda^{-k}(\lambda-1)^{k}\,\mathrm{d}\lambda, \tag{33}$$

and

$$r_k-r_{k\delta}=\int_{|\lambda|\geqslant\frac{1}{\delta}}\frac{|\lambda|^{n}_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}}\lambda^{-k}(\lambda-1)^{k}|\lambda-1|^{-n}\mathrm{d}\lambda. \tag{34}$$

When $|z|>3R$, we have to consider only terms of the form $\phi(y)\psi(z+(\lambda-1)(z-y))$. For this to be nonzero, we must have $|y|\leqslant R$ and $|z+(\lambda-1)(z-y)|\leqslant R$. These inequalities imply

$$|\lambda||z|=|\lambda z|\leqslant|\lambda z+(1-\lambda)y|+|(\lambda-1)y|\leqslant(1+|\lambda-1|)R,$$

and since $|z|>3R$ we obtain

$$3|\lambda|\leqslant(1+|\lambda-1|).$$

But this is true only when $-1\leqslant\lambda\leqslant\frac{1}{2}$ and our region of integration excludes this. Hence $A_k(z)=A_{k,\delta}(1,z)$ when $|z|>3R$ follows. Next assuming $|z|\leqslant 3R$, to estimate (31), we apply mean value to theorem to get an estimate

$$|\psi(z+(\lambda-1)(z-y))-\psi(z)|\leqslant c|\lambda-1||z-y|.$$

Thus the integral (31) is dominated by

$$c\int_{|y|\leqslant R}\int_{1/(1+\delta)}^{1/(1-\delta)}|\lambda|^{n-k}|\lambda-1|^{k}|z-y|\,\mathrm{d}\lambda\,\mathrm{d}y\leqslant c\delta$$

since $(\frac{1}{1+\delta},\frac{1}{1-\delta})\subset(\frac{2}{3},2)$ and the integrand is bounded there.

Coming to (32), we have, again by an application of mean value theorem,

$$|A_{2,k}(z)-c_{k,\delta}(1,z)|$$

$$\leqslant c\int_{|\lambda|\geqslant\frac{1}{\delta}}\int|\lambda|^{n-k}|\lambda-1|^{k-1}|\psi(z+(\lambda-1)(z-y))||z-y|\,\mathrm{d}\lambda\,\mathrm{d}y$$

$$\leqslant c\int_{|\lambda-1|\geqslant\frac{1}{2\delta}}\int|\lambda-1|^{n-1}|\psi(z+(\lambda-1)(z-y))||z-y|\,\mathrm{d}\lambda\,\mathrm{d}y.$$

Putting $(\lambda-1)|z-y| = \lambda'$ and writing $|z-y|^{-1}(z-y) = w$, we obtain

$$|A_{2,k}(z) - c_{k,\delta}(1, z)|$$

$$\leqslant c \int_{|\lambda'| \geqslant |z-y|/2\delta} \int |\lambda'|^{n-1} |z-y|^{1-n} |\psi(z+\lambda' w)| \, d\lambda' \, dy.$$

Note that $|\psi(z+\lambda' w)| = 0$ if $|\lambda'| \geqslant 4R$. So we have to integrate only over $|\lambda'| \leqslant 4R$. This gives

$$|A_{2,k}(z) - c_{k,\delta}(1, z)| \leqslant c \int_{|z-y| \leqslant 8R_\delta} |z-y|^{1-n} \, dy \int_{|\lambda| \leqslant 4R} |\lambda|^{n-1} \, d\lambda \leqslant c\delta.$$

It is easy to check that $|h_k - h_{k,\delta}| \leqslant c\delta$ and for the remaining term we need to consider only the case when $\varepsilon_2 = 1$ and in that case it is simple to estimate it in terms of δ. This completes step (2).

Step 3. Next we write

$$A_{k,\delta}(1, z) = A_{k,\delta k+1}(1, z) + D_{k,\delta}(1, z), \tag{35}$$

with

$$D_{k,\delta}(1, z) = A_{k,\delta}(1, z) - A_{k,\delta k+1}(1, z).$$

We have to consider the convolutions of f with the kernels defined by

$$E_{k,\delta}(z_0, z) = |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} \{A_{k,(\delta|z_0|^{-1})k+1}(1, z_0^{-1} z) - A_k(z_0^{-1} z)\} \quad \text{for} \quad |z_0| \geqslant 2\delta \tag{36}$$

$$= 0, \quad \text{for} \quad |z_0| < 2\delta,$$

$$A_k^\delta(z_0, z) = |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} A_k(z_0^{-1} z), \quad \text{for} \quad |z_0| \geqslant 2\delta \tag{37}$$

$$= 0, \quad \text{for} \quad |z_0| < 2\delta,$$

$$D_k^\delta(z_0, z) = |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} D_{k,\delta|z_0|^{-1}}(1, z_0^{-1} z), \quad \text{for} \quad |z_0| \geqslant 2\delta \tag{38}$$

$$= 0, \quad \text{for} \quad |z_0| < 2\delta.$$

We will take up the terms one by one. For the first term (36), we claim

$$\lim_{\delta \to 0} \iint E_{k,\delta}(z_0, z) f(x_0 - z_0, x - z) \, d\lambda \, dz = 0. \tag{39}$$

By step (2), we see that

$$|E_{k,\delta}(z_0, z)| \leqslant c\delta |z_0|^{-n-2} \text{ and } E_{k,\delta}(z_0, z) = 0$$

if $|z| \geqslant 3R|z_0|$ or $|z_0| < 2\delta$. Defining an integrable function L_k by

$$L_k(z_0, z) = c|z_0|^{-n-2}, \quad \text{for} \quad |z| \leqslant 3R|z_0| \text{ and } |z_0| \geqslant 2 \tag{40}$$

$$= 0, \quad \text{for } |z| > 3R|z_0| \text{ or } |z_0| < 2.$$

We observe that

$$|E_{k,\delta}(z_0, z)| \leqslant \delta^{-n-1} L_k(\delta^{-1} z_0, \delta^{-1} z).$$

Therefore, we get

$$\lim_{\delta\to 0}\iint E_{k,\delta}(z_0, z) f(x_0 - z_0, x - z)\,dz_0\,dz$$

$$= (\lim_{\delta\to 0}\iint E_{k,\delta}(z_0, z)\,dz_0\,dz) f(x_0, x).$$

Write

$$E_{k,\delta}(z_0, z) = |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k}[A_{k,(\delta|z_0|^{-1})k+1}(1, z_0^{-1}z) - A_k(z_0^{-1}z)]$$

$$= |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} F_{\delta|z_0|^{-1}}(1, z_0^{-1}z),$$

with

$$F_\delta(1, z) = \int_{1/(1+\delta^{k+1})}^{1/(1-\delta^{k+1})}\int \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}}\lambda^{-k}(\lambda-1)^k y^\alpha\phi(y)\psi(\lambda z-(\lambda-1)y)\,d\lambda\,dy$$

$$+ \int_{|\lambda|\geq\delta^{-(k+1)}}\frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}}\lambda^{-k}(\lambda-1)^k y^\alpha\phi(y)\psi(\lambda z-(\lambda-1)y)d\lambda dy$$

$$= F^1_\delta(1, z) + F^2_\delta(1, z) \text{ (say)}.$$

Now we have

$$\int F^1_{\delta|z_0|^{-1}}(1, z_0^{-1}z)dz = |z_0|^n(\int y^\alpha\phi)(\int\psi)\int_{(1+(|z_0|^{-1}\delta)^{k+1})^{-1}}^{(1-(\delta|z_0|^{-1})^{k+1})^{-1}}$$

$$|\lambda|^0_{\varepsilon_1+\varepsilon_2}\lambda^{-k}\frac{(\lambda-1)^k}{|\lambda-1|_{\varepsilon_1}}d\lambda$$

and we obtain

$$\left|\int\int_{|z_0|\geq 2\delta}|z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} F^1_{\delta|z_0|^{-1}}(1, z_0^{-1}z)\,dz_0\,dz\right|.$$

$$\leq c\int_{|z_0|\geq 2\delta}|z_0|^{k-1}dz_0\int_{(1+(\delta|z_0|^{-1})^{k+1})^{-1}}^{(1-(\delta|z_0|^{-1})^{k+1})^{-1}}\lambda^{-k}|\lambda-1|^{k-1}d\lambda$$

$$\leq c\delta^k\int_{|z_0|\geq 2}|z_0|^{-2}dz_0 \leq c\delta^k.$$

For F^2_δ term we add and subtract the term $z^\alpha\phi(z)$ and both terms can be estimated a in the previous case to yield a bound of δ^k. Thus we obtain

$$\lim_{\delta\to 0}\iint E_{k,\delta}(z_0, z)\,dz_0\,dz = 0$$

and this proves the claim.

Regarding (37), we claim the following:

$$\lim_{\delta\to 0}\int\int_{|z_0|\geq 2\delta}|z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k}A_k(z_0^{-1}z)f(x_0 - z_0, x - z)dz_0\,dz = A_k f,$$

where

$$A_k f(x_0, x) = \iint |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} A_k(z_0^{-1}z) f(x_0 - z_0, x - z)\, dz_0\, dz,$$

and A_k defines a bounded linear operator from L^p into L^q for $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{k}{n+1}$.

We first make the observation that

$$|A_k(z)| \leqslant c(1 + |z|^2)^{-n/2}.$$

Indeed, we have to prove this only for $A_{k,\delta}(1, z)$ since

$$|A_k(z) - A_{k,\delta}(1, z)| \leqslant c\delta \quad \text{for} \quad |z| \leqslant 3R$$

and they are equal when $|z| > 3R$. From the expression

$$A_{k,\delta}(1, z) = \iint_{T_\delta} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda - 1|_{\varepsilon_1}} \lambda^{-k}(\lambda - 1)^k y^\alpha \phi(y) \psi(z + (\lambda - 1)(z - y))\, d\lambda\, dy,$$

it is clear that $A_{k,\delta}(1, z)$ is bounded and for $|z| > 3R$ the integrand vanishes unless $|\lambda||z| \leqslant (1 + |\lambda - 1|)R$ and $-1 \leqslant \lambda \leqslant \frac{1}{2}$. Thus in the above, the λ integration is only over the interval $|\lambda| \leqslant \frac{3R}{|z|}$ and this gives the required estimate.

Now

$$A_k^\delta(z_0, z) = |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} A_k(z_0^{-1}z)$$

is homogeneous of degree $-n-1+k$. By setting $\lambda^2 = |z_0|^2 + |z|^2$ we have

$$A_k^\delta\left(\frac{z_0}{\lambda}, \frac{z}{\lambda}\right) = \lambda^{(n+1-k)} |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} A_k(z_0^{-1}z)$$

which shows that

$$\left|A_k^\delta\left(\frac{z_0}{\lambda}, \frac{z}{\lambda}\right)\right| \leqslant \left(1 + \left|\frac{z}{z_0}\right|^2\right)^{\frac{n+1-k}{2}} |A_k(z_0^{-1}z)|.$$

$$\leqslant c$$

Therefore

$$|A_k^\delta(z_0, z)| \leqslant c(|z_0|^2 + |z|^2)^{\frac{-n-1+k}{2}},$$

which is the Riesz-potential of order $n+1-k$. Hence the convolution with $A_k^\delta(z_0, z)$ makes sense and we can pass to the limit as $\delta \to 0$. Also

$$A_k f(x_0, x) = \iint |z_0|^{-n-1+k}_{\varepsilon_1+\varepsilon_2+k} A_k(z_0^{-1}z) f(x_0 - z_0, x - z)\, dz_0\, dz$$

defines a bounded linear operator from L^p into L^q whenever $1 < p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{k}{n+1}$.

Q.E.D.

Finally for the convolution with $D_k^\delta(z_0, z)$ we have the following claim.

$$\lim_{\delta \to 0} \iint D_k^\delta(z_0, z) f(x_0 - z_0, x - z) dz_0\, dz = 0$$

in the L^p norm for any p satisfying

$$\frac{n+1}{n+2-k} < p < \frac{n+1}{n+1-k}.$$

Proof. We will show that for any p satisfying the above inequalities D_k^δ is in $L^p(R^{n+1})$ and the L^p norm of it tends to zero as $\delta \to 0$ and this will prove the claim.

We first make the observation that $D_k^\delta(z_0, z)$ vanishes for $|z| > 3R|z_0|$. This follows from the fact that

$$D_{k,\delta}(1, z) = A_{k,\delta}(1, z) - A_{k,\delta k+1}(1, z),$$

and this vanishes for $|z| > 3R$. Thus we have

$$\iint |z_0|^{-p(n+1-k)} |D_{k,\delta|z_0|^{-1}}(1, z_0^{-1} z)|^p dz_0\, dz$$

$$\leqslant \int_{|z| \leqslant 3R|z_0|} \delta^p \int_{|z_0| \geqslant 2\delta} |z_0|^{-p(n+1-k)} |z_0|^{-p} dz_0\, dz$$

$$\leqslant c\delta^{n+1-p(n+1-k)} \int_{|z_0| \geqslant 2} |z_0|^{n-p(n+2-k)} dz_0.$$

Since $n - p(n+2-k) < -1$ the integral is finite and since $n+1-p(n+1-k) > 0$ the L^p norm of D_k^δ tends to 0.

Q.E.D.

Step 4. From step (1), (2) and (3), the proof of the Proposition (2.1) is complete save for the expression for A_k. We now write $A_k(z)$ in the form

$$A_k(z) = \lim_{\delta \to 0} \left\{ \iint_{\delta \leqslant |\lambda| \leqslant \delta^{-1}} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k y^\alpha \phi(y) \psi(z + (\lambda-1)(z-y)) d\lambda dy \right.$$

$$+ \left[h_k - \int_{\delta \leqslant |\lambda| \leqslant 2} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k d\lambda \right] (\int y^\alpha \phi) \psi(z)$$

$$\left. + \left[r_k - \int_{2 \leqslant |\lambda| \leqslant \delta^{-1}} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k |\lambda-1|^{-n} d\lambda \right] z^\alpha \phi(z) (\int \psi) \right\}.$$

Since we have removed the singularities at 0 and ∞ the above integral exists and the bracketed expressions tend to zero as $\delta \to 0$. Thus the above limit exists in the L^2 norm

and equals

$$A_k(z) = \lim_{\delta \to 0} \int \int_{\delta \leq |\lambda| \leq \delta^{-1}} \frac{|\lambda|^n_{\varepsilon_1 + \varepsilon_2}}{|\lambda - 1|_{\varepsilon_1}} \lambda^{-k} (\lambda - 1)^k y^\alpha \phi(y) \psi(z + (\lambda - 1)(x - y)) \mathrm{d}\lambda \, \mathrm{d}y.$$

This completes the proof of Proposition 2.1.

Now it remains to consider convolution with kernels of the form

$$\int_0^1 (1-t)^m A_{\alpha^*, \delta, t}(x_0, x, z_0, z) \, \mathrm{d}t \text{ with } |\alpha^*| = m + 1.$$

Concerning the convolution with

$$A_{\alpha^*, \delta, t}(x_0, x, z_0, z),$$

we will prove now the following Lemma. For the sake of convenience, we will drop t and write $G_{m+1, \delta}$ instead of $A_{\alpha^*, \delta, t}$. With this notation, we have

LEMMA 2.4.

$$G_{m+1} f(x_0, x) = \lim_{\delta \to 0} \iint G_{m+1, \delta}(x_0, x, z_0, z) f(x_0 - z_0, x - z) \, \mathrm{d}z_0 \, \mathrm{d}z \tag{41}$$

exists and we obtain

$$G_{m+1} f(x_0, x) = \iint K_{m+1}(x_0, x, z_0, z) f(x_0 - z_0, x - z) \, \mathrm{d}z_0 \, \mathrm{d}z, \tag{42}$$

where the Kernel $K_{m+1}(x_0, x, z_0, z)$ is given by

$$K_{m+1}(x_0, x, z_0, z) = \lim_{\delta \to 0} \int \int_{\delta \leq |z_0 - y_0| \leq \delta^{-1}} |y_0|^{-n-1}_{\varepsilon_1} |z_0 - y_0|^{-n-1}_{\varepsilon_2}$$

$$y_0^{\alpha_0} y^\alpha (D^{\alpha^*} b)(x_0 - t y_0, x - t y) \phi(y_0^{-1} y) \psi\left(\frac{z - y}{z_0 - y_0}\right) \mathrm{d}y_0 \, \mathrm{d}y. \tag{43}$$

The limit exists in the L^2 norm and G_{m+1} defines a bounded linear operator from L^p into L^q whenever $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{m}{n+1}$ and the norm of G_{m+1} is independent of t.

Proof. Let us write g instead of $(D^{\alpha^*} b)$ and define

$$H_\delta(x_0, x, y_0, y) = y_0^{\alpha 0} y^\alpha |y_0|^{-n-1}_{\varepsilon_1} \phi(y_0^{-1} y) g(x_0 - t y_0, x - t y), \text{ for } |y_0| \geq \delta$$

$$= 0, \quad \text{otherwise.}$$

In the integral

$$(G_{m+1,\delta} * f)(x_0, x)$$

$$= \iint f(x_0 - z_0, x - z)\,dz_0\,dz \int\limits_{\substack{|y_0| \geqslant \delta \\ |z_0 - y_0| \geqslant \delta}} y_0^{\alpha_0} y^{\alpha} |y_0|_{\varepsilon_1}^{-n-1} \phi(y_0^{-1} y)$$

$$|z_0 - y_0|_{\varepsilon_2}^{-n-1} g(x_0 - ty_0, x - ty) \psi\left(\frac{z-y}{z_0 - y_0}\right) dy_0\,dy$$

changing the order of integration we have

$$(G_{m+1,\delta} * f)(x_0, x) = \iint H_\delta(x_0, x, y_0, y)\, K_2^\delta f(x_0 - y_0, x - y)\,dy_0\,dy$$

where $K_2^\delta f$ is the integral given by

$$(K_2^\delta f)(x_0, x) = \int\limits_{|z_0| \geqslant \delta} \int |z_0|_{\varepsilon_2}^{-n-1} \psi(z_0^{-1} z) f(x_0 - z_0, x - z)\,dz_0\,dz.$$

Let us now estimate the L^2 norm of $H_\delta(x_0, x, y_0, y)$ for fixed (x_0, x).

$$\iint |H_\delta(x_0, x, y_0, y)|^2\,dy_0\,dy$$

$$\leqslant \iint |y_0|^{2(m-n)} |\phi(y_0^{-1} y)|^2 |g(x_0 - ty_0, x - ty)|^2\,dy_0\,dy$$

$$\leqslant \operatorname*{Sup}_{(x_0, x)} (1 + |x_0|^2 + |x|^2)^2 |g(x_0, x)|^2 \iint |y_0|^{2(m-n)} |\phi(y_0^{-1} y)|^2$$

$$(1 + |x_0 - ty_0|^2)^{-2}\,dy_0\,dy$$

$$\leqslant c(1 + |x_0|^2)^2 t^{2(n-m)-n-1} \|\phi\|_2^2 \int |y_0|^{2(m-n)+n} (1 + |y_0|^2)^{-2}\,dy_0.$$

Since we have chosen $m = [(n+1)/2]$ the integral converges. Thus we see that H_δ is in L^2 and in the L^2 norm it converges to the function H defined by

$$H(x_0, x, y_0, y) = y_0^{\alpha_0} y^{\alpha} |y_0|_{\varepsilon_1}^{-n-1} \phi(y_0^{-1} y) g(x_0 - ty_0, x - ty).$$

Since $K_2^\delta f$ is a truncated singular integral, it converges boundedly to the function $K_2 f$ in the L^2 norm. Therefore, passing to the limit as $\delta \to 0$, we obtain

$$G_{m+1} * f = \iint H(x_0, x, y_0, y) K_2 f(x_0 - y_0, x - y)\,dy_0\,dy.$$

Now suppose m_2 is the multiple associated with the singular integral $K_2 = S(\psi, \varepsilon_2)$. Then it is shown in [2] that we have

$$m_2(z_0, z) = \lim_{\delta \to 0} \int\limits_{\delta \leqslant |x_0| \leqslant \delta^{-1}} \int \exp[-i(x_0 z_0 + x \cdot z)] |x_0|_{\varepsilon_2}^{-n-1} \psi(x_0^{-1} x)\,dx_0\,dx$$

$$= \lim_{\delta \to 0} m_2^\delta(z_0, z),$$

where m_2^δ is the Fourier transform of the function which equals

$$|x_0|_{\varepsilon_2}^{-n-1}\psi(x_0^{-1}x) \quad \text{for} \quad \delta \leqslant |x_0| \leqslant \delta^{-1}$$

and zero elsewhere. The relation between K_2 and m_2 is given by $(K_2 f)\hat{} = m_2 \hat{f}$ where $\hat{}$ denotes the Fourier transformation.

In the expression for $G_{m+1} * f$ making the change of variable $y_0 \to y_0 + x_0$, $y \to y + x$, writing $K_2 f$ in terms of $m_2 \hat{f}$ and changing the order of integration, we obtain

$$G_{m+1} * f(x_0, x) = \iint \exp[i(z_0 x_0 + z \cdot x)] m_2(z_0, z) \hat{H}(x_0, x, z_0, z) \hat{f}(z_0, z) \mathrm{d}z_0 \mathrm{d}z$$

the Fourier transform being in the z variables. Writing

$$F^{-1}[m_2 \hat{H}] = K_{m+1}(x_0, x, z_0, z)$$

we obtain the formula

$$G_{m+1} * f(x_0, x) = \iint K_{m+1}(x_0, x, z_0, z) f(x_0 - z_0, x - z) \mathrm{d}z_0 \mathrm{d}z.$$

Since m_2 is a bounded function and $\hat{H}(x_0, x, z_0, z)$ is an L^2 function K_{m+1} is an L^2 function in the z variables. As $m_2^\delta \to m_2$ as $\delta \to 0$, we get

$$K_{m+1}(x_0, x, z_0, z) = \lim_{\delta \to 0} F^{-1}[m_2^\delta \hat{H}](z_0, z).$$

But

$$F^{-1}[m_2^\delta \hat{H}](z_0, z) = (F^{-1} m_2^\delta * \hat{H})(z_0, z)$$

$$= \int_{\delta \leqslant |y_0| \leqslant \delta^{-1}} \int |y_0|_{\varepsilon_2}^{-n-1} \psi(y_0^{-1} y) H(x_0, x, z_0 - y_0, z - y) \mathrm{d}y_0 \mathrm{d}y$$

$$= \int_{\delta \leqslant |z_0 - y_0| \leqslant \delta^{-1}} \int |y_0 - z_0|_{\varepsilon_2}^{-n-1} \psi\left(\frac{z - y}{z_0 - y_0}\right) H(x_0, x, y_0, y) \mathrm{d}y_0 \mathrm{d}y.$$

Thus we have obtained

$$K_{m+1}(x_0, x, z_0, z) = \lim_{\delta \to 0} \int_{\delta \leqslant |z_0 - y_0| \leqslant \delta^{-1}} \int |y_0 - z_0|_{\varepsilon_2}^{-n-1} |y_0|_{\varepsilon_1}^{-n-1} y_0^\alpha y^\alpha$$

$$\phi(y_0^{-1} y) g(x_0 - t y_0, x - t y) \psi\left(\frac{z - y}{z_0 - y_0}\right) \mathrm{d}y_0 \mathrm{d}y.$$

Since we have an integral in t variable also and since the L^2 norm of H depends on t in a non-integrable way, we are unable to assert that

$$\int_0^1 (1 - t)^m G_{m+1} * f \, \mathrm{d}t$$

defines a bounded linear operator on L^2. Instead, we have

$$|G_{m+1} * f| \leqslant c \iint |y_0|^{(-n+m)} |\phi(y_0^{-1} y)| |K_2 f(x_0 - y_0, x - y)| \mathrm{d}y_0 \mathrm{d}y$$

and the kernel

$$|y_0|^{-n+m} |\phi(y_0^{-1} y)|$$

is homogeneous of degree $-n+m$ and hence $G_{m+1}*f$ defines a bounded linear operator from L^p into L^q whenever

$$1 < p < q < \infty, \frac{1}{q} = \frac{1}{p} - \frac{m+1}{n+1}$$

with the norm independent of t.

Q.E.D.

Proof of the theorem:

Combining Lemmas 2.3, 2.4 and Proposition 2.1 together, we immediately obtain the Theorem stated in the introduction.

3. Some estimates

To obtain the composition formula, we have made extremely restrictive assumptions on ϕ and ψ viz ϕ, ψ are in $D(R^n)$. We want now to relax these conditions by getting appropriate estimates.

As far as the term A is concerned, we immediately obtain the following theorem which is proved in [3].

THEOREM 3.1.
Let B denote the Banach space of all functions f for which the norm

$$\|f\|_B^2 = \int_{R^n} |f(x)|^2 (1+|x|^2)^{\frac{n+1}{2}} \, dx < \infty,$$

and let B_0 denote the subspace of B containing those functions satisfying $\int f(x)\,dx = 0$. Then

$$\| A \|_B \leqslant c \| \phi \|_B \| \psi \|_B \text{ for all } \phi, \psi \in D(R^n)$$

satisfying $\int \phi = 0$, $\int \psi = 0$, so the mapping $(\phi, \psi) \to A$ extends to a bounded bilinear mapping from $B_0 \times B_0$ to B_0.

We are now going to show that similar estimates for the terms A_k can be obtained. We are not venturing to get optimal estimates. We just imitate the proof of the above theorem and we would not give a complete proof. We will just give a sketch of the proof.

To start with, we first prove the following Lemma concerning the Fourier transform of A_k.

LEMMA 3.1.
Under the hypotheses of the main theorem, we have

$$\hat{A}_k(\xi) = \lim_{\delta \to 0} \int_{\delta \leqslant |1-s| \leqslant \delta^{-1}} |1-s|_{\varepsilon_2}^{-1} |s|_{\varepsilon_1}^{-1} \, s^k (D^\alpha \hat{\phi})(s\xi) \hat{\psi}((1-s)\xi) \, ds$$

Proof. Since we have

$$A_k(z) = \lim_{\delta \to 0} \int \int_{\delta \leqslant |\lambda| \leqslant \delta^{-1}} \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k y^\alpha \phi(y)\psi(z+(\lambda-1)(z-y))\mathrm{d}\lambda \mathrm{d}y$$

and the limit exists in the L^2 sense we can interchange the limit and the Fourier transform. Thus

$$\hat{A}_k(\xi) = \lim_{\delta \to 0} \int \int_{\delta \leqslant |\lambda| \leqslant \delta^{-1}} \exp(-i\xi \cdot z) \frac{|\lambda|^n_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k y^\alpha \phi(y)$$
$$\psi(z+(\lambda-1)(z-y))\,\mathrm{d}\lambda\,\mathrm{d}y.$$

The triple integral is absolutely convergent and so we may put $z' = z+(\lambda-1)(z-y)$ and then integrate with respect to z' and y to obtain

$$\hat{A}_k(\xi) = \lim_{\delta \to 0} \int_{\delta \leqslant |\lambda| \leqslant \delta^{-1}} \frac{|\lambda|^0_{\varepsilon_1+\varepsilon_2}}{|\lambda-1|_{\varepsilon_1}} \lambda^{-k}(\lambda-1)^k \hat{\psi}\left(\frac{\xi}{\lambda}\right)(D^\alpha \hat{\phi})\left(\frac{\lambda-1}{\lambda}\xi\right)\mathrm{d}\lambda.$$

Now the change of variable $(\lambda-1)/\lambda = s$ gives

$$\hat{A}_k(\xi) = \lim_{\delta \to 0} \int_{\delta \leqslant |1-s| \leqslant \delta^{-1}} |1-s|^{-1}_{\varepsilon_2} |s|^{-1}_{\varepsilon_1} s^k (D^\alpha \hat{\phi})(s\xi)$$
$$\hat{\psi}((1-s)\xi)\,\mathrm{d}s.$$

Q.E.D.

We will now prove the following theorem.

THEOREM 3.2.
For $1 \leqslant k \leqslant m$ let B_k denote the space of all functions f for which the norm

$$\|f\|^2_{B_k} = \int_{R^n} |f(x)|^2 (1+|x|^2)^{\frac{n+1}{2}-k} < \infty.$$

Then $\|A_k\|_{B_k} \leqslant c\|\phi\|_B \|\psi\|_B$ for all $\phi, \psi \in D(R^n)$ satisfying $\int \phi = 0$ and so the mapping $(\phi, \psi) \to A_k$ extends to a bounded bilinear mapping from $B_0 \times B_0$ into B_k.

Proof. As we have remarked earlier, the proof is exactly same as the proof of Theorem 2.6 of [3]. We remark that $f \in B_k$ is equivalent to

$$\hat{f} \in H^{\frac{n+1}{2}-k}$$

and the condition $\int f = 0$ is equivalent to $\hat{f}(0) = 0$. Thus we have to obtain an estimate of the form

$$\|\hat{A}_k\|_{\frac{n+1}{2}-k} \leqslant c\|\hat{\phi}\|_{\frac{n+1}{2}} \|\hat{\psi}\|_{\frac{n+1}{2}}.$$

Again we have to consider two cases.

Case 1: $\dfrac{n+1}{2}$ is an integer.

In this case the $H^{\frac{n+1}{2}-k}$ norm is equivalent to

$$\left(\sum_{|\alpha|\leqslant\frac{n+1}{2}-k}\|D^{\alpha}f\|_2^2\right)^{1/2}$$

and we have to bound each of $\|D^{\alpha}\hat{A}_k\|_2$ by the product of $H^{\frac{n+1}{2}}$ norms of $\hat{\phi}$ and $\hat{\psi}$. Our hypotheses allow us to differentiate $\hat{A}_k$ under the integral sign (and even we can put $\delta=0$ since $\hat{\phi}(0)=0=\hat{\psi}(0)$ the integral being absolutely convergent). Thus $D^{\beta}\hat{A}_k$ is a sum of terms of the form

$$\int_{-\infty}^{\infty}|1-s|^{-1+|\gamma|}_{\varepsilon_2+|\gamma|}|s|^{-1+|\nu|}_{\varepsilon_1+|\nu|}s^k(D^{\alpha+\nu}\hat{\phi})(s\xi)(D^{\gamma}\hat{\psi})((1-s)\xi)\,ds$$

where $\nu+\gamma=\beta$.

Noting that $D^{\gamma}\hat{\psi}\in H^{\frac{n+1}{2}-|\gamma|}$ and $D^{\alpha+\nu}\hat{\phi}\in H^{\frac{n+1}{2}-k-|\nu|}$ we need to show

$$\left\|\int_{-\infty}^{\infty}|s|^{b-1}|1-s|^{a-1}f((1-s)\xi)g(s\xi)\,ds\right\|_2$$

$$\leqslant c\|f\|_{\frac{n+1}{2}-a}\|g\|_{\frac{n+1}{2}-b}$$

provided $a+b\leqslant\dfrac{n+1}{2}$ and $f(0)=0$ if $a=0$. In fact it is sufficient to prove this for $a+b=\dfrac{n+1}{2}$ and $a=0$, $b=k$ and the proof given in [3] goes without any change.

Case 2. $\dfrac{n+1}{2}$ is not an integer.

In this case the $H^{\frac{1}{2}(n+1)-k}$ norm is equivalent to

$$\left(\|f\|_2^2+\sum_{|\alpha|=\frac{n}{2}-k}\iint|D^{\alpha}f(x)-D^{\alpha}f(y)|^2\frac{dx\,dy}{|x-y|^{n+1}}\right)^{1/2}.$$

In this case we need to obtain estimates of the form

$$\iint\left|\int_{-\infty}^{\infty}|s|^{b-1}|1-s|^{a-1}[f((1-s)x)g(sx)-f((1-s)y)g(sy)]\,ds\right|^2\frac{dx\,dy}{|x-y|^{n+1}}$$

$$\leqslant c\|f\|^2_{\frac{n+1}{2}-a}\|g\|_{\frac{n+1}{2}-b}$$

again with $a+b=n/2$ and $f(0)=0$ if $a=0$. The proof of this goes along similar lines with that of Theorem 2.6 of [3].

Q.E.D.

4. Remarks and comments

(1) Having obtained the above estimates, now we see that A with $\phi, \psi \in B_0$ defines a singular integral operator $S(A, \varepsilon_1 + \varepsilon_2)$. But what about A_k's? The conditions A_k belongs to B_k is equivalent to the condition that the restriction of the kernel

$$|z_0|_{\varepsilon_1+\varepsilon_2+k}^{-n-1+k} A_k(z_0^{-1}z)$$

to the unit sphere S^n is square integrable. A close look at the proof of the boundedness of the convolutions with Riesz potentials in [2] reveals that with the condition A_k belongs B_k the kernels

$$|z_0|_{\varepsilon_1+\varepsilon_2+k}^{-n-1+k} A_k(z_0^{-1}z)$$

indeed defines bounded linear operators between L^p and L^q for appropriate p and q. Again for the convolution with G_{m+1} the above conditions on ϕ and ψ do imply that $G_{m+1} * f$ is bounded from L^p into L^q. Thus our composition formula is valid even when $\phi, \psi \in B_0$.

(2) The same theorem holds even if we replace the kernels by more general kernels of the form $|z_0|_\varepsilon^{-n-1}\Phi(x_0, x, z_0^{-1}z)$ where Φ satisfies the following conditions.

(i) $|D^{\alpha^*}\Phi(x_0, x, z)| \leqslant c|\phi_{\alpha^*}(z)|$ for $|\alpha^*| \leqslant m = \left[\frac{n+1}{2}\right]$,

(ii) $|D^{\alpha^*}\Phi(x_0, x, z)| \leqslant c|\phi_{\alpha^*}(z)|(1+|x_0|^2+|x|^2)^{-1}$ for $|\alpha^*| = m+1$,

with the functions ϕ_{α^*} belonging to $D(R^n)$.

3. The estimates we got are not optimal. Further, we have to go in the Taylor expansion of b upto derivatives of order $m+1$. We do not know whether $1 + \left[\frac{n+1}{2}\right]$ is optimal or not.

References

[1] Calderon A P and Zygmund 1956 On singular integrals, *Am. J. Math.* **78** 289–309

[2] Stein E M 1970 *Singular integrals and differentiability properties of functions* (Princeton: University Press)

[3] Strichartz R S 1982 Compositions of singular integral operators, *J. Funct. Anal.* **49** 91–127

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The homomorphism $M^* \otimes N \to \mathrm{Hom}(M,N)$

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Abstract. In this note we prove two theorems. In theorem 1 we prove that if M and N are two non-zero reflexive modules of finite projective dimensions over a Gorenstein local ring, such that $\mathrm{Hom}\,(M, N)$ is a third module of syzygies, then the natural homomorphism $M^* \otimes N \to \mathrm{Hom}\,(M, N)$ is an isomorphism. This extends the result in [7]. In theorem 2, we prove that projective dimension of a module M over a regular local ring R is less than or equal to n if and only if $\mathrm{Ext}^n_R(M, R) \otimes M \to \mathrm{Ext}^n_R(M, M)$ is surjective; in which case it is actually bijective. This extends the usual criterion for the projectivity of a module.

Keywords. Homomorphism; Gorenstein local ring; isomorphism; projectivity.

Throughout, R will denote a commutative noetherian local ring with identity. All modules will be assumed to be finitely generated and unitary.

Auslander and Goldman [2] in their study of orders in simple algebras over the quotient field of a regular local ring R, discovered that if M is a reflexive R-module such that $\mathrm{Hom}_R(M, M)$ is free, then M itself is free. This is generalized in [7] to assert if M and N are non-zero reflexive modules over R, a regular local ring, such that $\mathrm{Hom}_R(M, N)$ is non-zero free, then so are M and N. This is done by showing that the natural homomorphism $M^* \otimes N \to \mathrm{Hom}\,(M, N)$ is an isomorphism. Here $*$ denotes duals. This can be improved further. This is theorem 1 of the present note. The projectivity of a module M is equivalent to the surjectivity of the homomorphism $M^* \otimes M \to \mathrm{Hom}(M, M)$. Theorem 2 is an extension of this giving a criterion for a module over a regular local ring to have projective dimension less than or equal to a given non-negative integer.

In what follows we shall abbreviate projective dimension as (p.d.), Krull dimension will be simply denoted as dim. All modules are zeroth syzygies. A module M is said to be an nth syzygy, for a positive integer n, if there is an exact sequence, $0 \to M \to F_{n-1} \to \ldots\ldots \to F_1 \to F_0$, where $F_0, F_1, \ldots$ are free modules. We shall use [8] as a reference for standard terms in Commutative Algebra. If $\ldots \to F_n \to F_{n-1} \ldots \to F_0 \to M \to O$ is a projective resolution of a module M, the kernal of $F_i \to F_{i-1}$ will be denoted as $\Omega^{i+1}M$, with the understanding that $F_{-1} = M$ and $\Omega^0 M = M$; the cokernal of $F_i^* \to F_{i+1}^*$ will be denoted as $\overline{D\Omega^i M}$. Although these definitions depend on the chosen projective resolution of M, they are well defined up to 'projective equivalence' (Auslander and Bridger [1]). Finally a module N is said to be 'rigid' if whenever $\mathrm{Tor}^R_1(M, N) = (0)$, for a module M, then $\mathrm{Tor}^R_j(M, N) = (0)$ for all $j \geqslant 1$. Over a regular local ring any module is 'rigid'. This is a theorem of by Lichtenbaum [5]. We now state the main theorem of this note:

THEOREM 1 Suppose M and N are non-zero reflexive modules of finite projective dimensions over a Cohen-Macaulay local ring R and N is rigid. Then if either $\mathrm{Ext}^1_R(\mathrm{Hom}(M, N)^*, R) = (0)$, or the ring R is Gorenstein and $\mathrm{Hom}(M, N)$ is a third module of syzygies, then the natural homomorphism $M^* \otimes N \to \mathrm{Hom}(M, N)$ is an isomorphism.

Remark. Let R be regular local and M, N reflexive R-modules. If $\mathrm{Hom}(M, N)$ is a free module, then $\mathrm{Ext}^1_R(\mathrm{Hom}(M, N)^*, R) = (0)$ so that the theorem 1 above implies the proposition in [7]. If $\mathrm{Ext}^1(\mathrm{Hom}(M, M), R) = (0)$, then since $\mathrm{Hom}(M, M) \cong \mathrm{Hom}(M^*, M^*)^*$, we get $\mathrm{Ext}^1(\mathrm{Hom}(M^*, M^*)^*, R) = (0)$ so that by the theorem above, the natural homomorphism $M^{**} \otimes M^* \to \mathrm{Hom}(M^*, M^*)$ is an isomorphism showing that M^* is free and so M is free. This is theorem 2.1 in [6]. The method of proving theorem 1 will be similar to that of [6] to some extent.

Proof of theorem 1 We first claim that the given conditions mean $\mathrm{Ext}^1_R(M, N) = 0$. This is proved by induction on dim R. Firstly, if q is a prime ideal of height at most two, then M_q being a reflexive R_q-module, depth M_q = depth R_q. Then the formula, P.d.M_q + depth M_q = depth R_q, implies M_q is R_q-free. Hence, $\mathrm{Ext}^1_{R_q}(M_q, N_q) = (0)$. We can therefore assume dim $R \geqslant 3$, and that $\mathrm{Ext}^1_{R_p}(M_p, N_p) = (0)$, for all non-maximal prime ideals p of R. Then either $\mathrm{Ext}^1_R(M, N) = 0$, or it is a non-zero module of finite length. Let

$$0 \to \mathrm{L} \to \mathrm{F} \to \mathrm{M} \to 0, \tag{1}$$

be an exact sequence with F free. Application of the functor Hom (, N) to (1) gives the exact sequence,

$$0 \to \mathrm{Hom}\,(\mathrm{M}, \mathrm{N}) \to \mathrm{Hom}\,(\mathrm{F}, \mathrm{N}) \to \mathrm{Hom}\,(\mathrm{L}, \mathrm{N}) \to \mathrm{Ext}^1\,(\mathrm{M}, \mathrm{N}) \to 0. \tag{2}$$

Now (2) can be broken into two short exact sequences

$$0 \to \mathrm{Hom}\,(\mathrm{M}, \mathrm{N}) \to \mathrm{Hom}\,(\mathrm{F}, \mathrm{N}) \to \mathrm{T} \to 0. \tag{3}$$

$$0 \to \mathrm{T} \to \mathrm{Hom}\,(\mathrm{L}, \mathrm{N}) \to \mathrm{Ext}^1\,(\mathrm{M}, \mathrm{N}) \to 0. \tag{4}$$

By the assumption on $\mathrm{Ext}^1(M, N)$, some power of maximal ideal of R annihilates $\mathrm{Ext}^1(M, N)$. Then by Cohen-Macaulay property of R, we find that grade $\mathrm{Ext}^1_R(M, N)$ = dim $R \geqslant 3$, *i.e.* $\mathrm{Ext}^j_R(\mathrm{Ext}^1_R(M, N), R) = (0)$ for $j = 0, 1, 2$. Therefore, dualizing (3) and (4), we get the following exact sequences

$$0 \to \mathrm{T}^* \to \mathrm{Hom}\,(\mathrm{F}, \mathrm{N})^* \to \mathrm{Hom}\,(\mathrm{M}, \mathrm{N})^* \to \mathrm{Ext}^1\,(\mathrm{T}, \mathrm{R}), \tag{5}$$

$$0 \to \mathrm{Hom}\,(\mathrm{L}, \mathrm{N})^* \to \mathrm{T}^* \to 0. \tag{6}$$

Putting (5) and (6) together we get the exact sequence.:

$$0 \to \mathrm{Hom}\,(\mathrm{L}, \mathrm{N})^* \to \mathrm{Hom}\,(\mathrm{F}, \mathrm{N})^* \to \mathrm{Hom}\,(\mathrm{M}, \mathrm{N})^* \to \mathrm{Ext}^1\,(\mathrm{T}, \mathrm{R}). \tag{7}$$

Now, if p is a prime ideal of height at most two, then M_p and N_p are free R_p-modules, as in the beginning of the proof. Hence (1) and (4) imply that T_p is R_p-free, *i.e.* $\mathrm{Ext}^1_R(T, R)_p = (0)$. Denoting by P, the image of the homomorphism $\mathrm{Hom}(M, N)^* \to \mathrm{Ext}^1(T, R)$, then by what precedes, $P_p = (0)$ *i.e.* if

$$\mathcal{C} = \text{annihilator of } P, \text{ then height } \mathcal{C} \geqslant 3.$$

Since R is Cohen Macaulay, grade P = height $\mathcal{C} \geqslant 3$.

Hence, we have $\mathrm{Ext}^i_R(P, R) = (0)$, for $i = 0, 1, 2$. Breaking (7) into two short exact sequences, as in the case of (2), we obtain

$$0 \to \mathrm{Hom}\,(\mathrm{L}, \mathrm{N})^* \to \mathrm{Hom}\,(\mathrm{F}, \mathrm{N})^* \to \mathrm{U} \to 0 \tag{8}$$

$$0 \to \mathrm{U} \to \mathrm{Hom}\,(\mathrm{M}, \mathrm{N})^* \to \mathrm{P} \to 0 \tag{9}$$

We dualize these sequences and make use of $\mathrm{Ext}^i(P, R) = (0)$ for $i = 0, 1$ in (9). Then putting together the resulting exact sequences we get,

$$0 \to \mathrm{Hom}\,(\mathrm{M}, \mathrm{N})^{**} \to \mathrm{Hom}\,(\mathrm{F}, \mathrm{N})^{**} \to \mathrm{Hom}\,(\mathrm{L}, \mathrm{N})^{**} \to \mathrm{Ext}^1\,(\mathrm{U}, \mathrm{R}) \tag{10}$$

From (9) and the facts that $\mathrm{Ext}^1\,(\mathrm{Hom}\,(M, N)^*, R) = (0)$ $\mathrm{Ext}^2\,(P, R) = (0)$, we get $\mathrm{Ext}^1\,(U, R) = (0)$, *i.e.* the sequence (10) becomes

$$0 \to \mathrm{Hom}\,(M, N)^{**} \to \mathrm{Hom}\,(F, N)^{**} \to \mathrm{Hom}\,(L, N)^{**} \to 0 \tag{11}$$

From (1), applying the functor Ext (, N), we get the exact sequence,

$$\mathrm{Hom}\,(F, N) \to \mathrm{Hom}\,(\mathrm{L}, \mathrm{N}) \to \mathrm{Ext}^1\,(\mathrm{M}, \mathrm{N}) \to 0.$$

Then with obvious maps, the diagram

$$\begin{array}{ccccc} \mathrm{Hom}\,(F, N) & \to & \mathrm{Hom}\,(L, N) & \to & \mathrm{Ext}^1\,(M, N) \to 0 \\ \downarrow & & \downarrow & & \\ \mathrm{Hom}\,(F, N)^{**} & \to & \mathrm{Hom}\,(L, N)^{**} & \to & 0 \end{array}$$

is commutative with isomorphic columns. Hence $\mathrm{Ext}^1_R(M, N) = (0)$. This proves our claim.

To conclude, we apply the Lemma [7] after noticing that only the rigidity of N is needed for its validity. We find that the natural homomorphism $M^* \otimes N \to \mathrm{Hom}\,(M, N)$ is an isomorphism.

THEOREM 2. Suppose R is regular local and M any non-zero R-module. For any non-negative integer n, $\mathrm{P.d.}M \leqslant n$ if and only if the natural homomorphism $\mathrm{Ext}^n_R(M, R) \otimes M \to \mathrm{Ext}^n_R(M, M)$ is surjective, in which case the map is actually bijective.

Remark If $n = 0$, this is a well known criterion for the projectivity of a module.

Proof Suppose $\mathrm{p.d.}M = j$. If $j < n$, clearly $\mathrm{Ext}^n_R(M, R) = (0) = \mathrm{Ext}^n_R(M, M)$. If $j = n$, then it is well known that the natural homomorphism

$$\mathrm{Ext}^n_R(M, R) \otimes M \to \mathrm{Ext}^n_R(M, M) \text{ is an isomorphism [1].}$$

Conversely, suppose the latter homomorphism is surjective, we make use of the fundamental exact sequences of Auslander and Bridger [1]:

$$\mathrm{Tor}^R_2\,(D\,\Omega^n M, N) \to \mathrm{Ext}^n_R(M, R) \otimes N \to \mathrm{Ext}^n_R(M, N) \to \mathrm{Tor}^R_1\,(D\,\Omega^n M, N) \to 0 \tag{12}$$

$$0 \to \mathrm{Ext}^1_R\,(D\,\Omega^n M, N) \to \mathrm{Tor}^R_n(M, N) \to \mathrm{Hom}\,(\mathrm{Ext}^n_R(M, R), N) \to \mathrm{Ext}^2_R\,(D\,\Omega^n M, N) \tag{13}$$

Taking $M = N$ in (12) and using the fact that the homomorphism $\mathrm{Ext}_R^n(M, R) \otimes N \to \mathrm{Ext}_R^n(M, N)$ is surjective, we get,

$\mathrm{Tor}_1^R(D\Omega^n M, M) = (0)$. By the rigidity of M, we get, $\mathrm{Tor}_j^R(D\Omega^n M, M) = (0)$, for $j \geqslant 1$. In particular $\mathrm{Tor}_n^R(D\Omega^n M, M) = (0)$. Then (13), yields $\mathrm{Ext}_R^1(D\Omega^n M, D\Omega^n M) = 0$. Applying [7], we find that $D\Omega^n M$ is projective, *i.e.* $\Omega^n M$ is projective. Hence, p.d.$M \leqslant n$.

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References

[1] Auslander M and Bridger M 1969 Stable module theory, *Am. Math. Soc.*, Providence, Rhode Island
[2] Auslander M and Goldman O 1960 Maximal orders. *Trans. Am. Math. Soc.* **97**
[3] Bass H 1963 on the ubiquity of Gorenstein rings. *Math. Z.* **82**
[4] Rees D 1957 The grade of an ideal or module. *Proc. Cambridge Philos. Soc.* 84
[5] Licheten baum S 1966 On the vanishing of Tor in regular local rings. *Ill. J. Math.* **10**
[6] Jinnah M I 1973 Reflexive modules over regular local rings. *Arch. Math.* **26**
[7] Jothilingam P 1975 A note on grade. *Nagoya Math. J.* **59**
[8] Serre J P 1965 Algebre Locale—Multiplicites. *Lecture Notes in Mathematics*, No. 11 (Berlin: Springer-Verlag)

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On the propagation of a multi-dimensional shock of arbitrary strength

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Abstract. In this paper the kinematics of a curved shock of arbitrary strength has been discussed using the theory of generalised functions. This is the extension of Moslov's work where he has considered isentropic flow even across the shock. The condition for a nontrivial jump in the flow variables gives the shock manifold equation (SME). An equation for the rate of change of shock strength along the shock rays (defined as the characteristics of the SME) has been obtained. This exact result is then compared with the approximate result of shock dynamics derived by Whitham. The comparison shows that the approximate equations of shock dynamics deviate considerably from the exact equations derived here. In the last section we have derived the conservation form of our shock dynamic equations. These conservation forms would be very useful in numerical computations as it would allow us to derive difference schemes for which it would not be necessary to fit the shock-shock explicitly.

Keywords. Nonlinear waves; shock dynamics; multi-dimensional shock propagation.

1. Introduction

It is now about three decades since serious attempts were made to solve the problem of the propagation of a shock of arbitrary shape and arbitrary strength. The main aim was to device a method to calculate the successive positions of a curved shock without going through the task of the calculation of the flow behind the shock. Whitham [7, 8] presented one such method, now known as shock dynamics, and showed that the motion of a curved shock depends on the initial shape and the distribution of the shock strength on the initial shock surface. The simplicity and elegance of Whitham's shock dynamics results from the fact that, in this theory, the influence of the waves which catch up with the shock from behind is accounted only approximately [5]. A precise and mathematically satisfying theory on the kinematics of a curved shock was first available in English in 1980 by the translation of the work of the Russian mathematician Maslov [2] who derived a first order partial differential equation satisfied by the shock manifold in (x_α, t)-space and then derived an infinite system of compatibility conditions on this manifold. These compatibility conditions are in the form of transport equations along shock rays. The partial differential equation for the shock (called shock manifold equation or SME) is not unique but it has been shown [3] that two different SMEs give the same set of shock rays starting from a given initial shock surface. Since Maslov was interested in establishing certain results for a weak shock, he considered the motion of an isentropic gas and thus paid no attention to the energy equation.

In this paper we consider the motion of a shock in an ideal gas with constant specific

heats with all the three conservation laws, namely the conservation laws representing balance of mass, momentum and energy. We use Maslov's method to derive the SME and the transport equation along a shock ray. Then we use our exact equations to discuss the correct form of the Whitham's shock dynamic equation. We show that shock dynamics not only incorrectly accounts for the effect of the waves which catch up with the shock from behind but the CCW approximation in the shock dynamics leads to a term whose coefficient differs significantly from the exact coefficient which we get using Maslov's method. Thus this paper shows the mathematical elegance of Maslov's method—the correct form of shock dynamics can be obtained by a systematic mathematical analysis without any assumption or approximation. Since Maslov's method is not widely known, we shall briefly indicate in the next section some of his important results using the theory of distribution.

2. Maslov's lemma on a conservation law

The conservation laws representing the three-dimensional motion of an ideal gas with constant specific heats are

$$\frac{\partial \rho}{\partial t} + \langle \nabla, \rho u \rangle = 0, \tag{1}$$

$$\frac{\partial (\rho u^i)}{\partial t} + \langle \nabla, \rho u u^i \rangle + \frac{\partial p}{\partial x_i} = 0 \qquad (i = 1, 2, 3), \tag{2}$$

and

$$\frac{\partial}{\partial t}\left\{\tfrac{1}{2}\rho \langle u, u \rangle + \frac{p}{(\gamma - 1)}\right\} + \langle \nabla, \tfrac{1}{2}\rho \langle u, u \rangle u + \frac{\gamma}{(\gamma - 1)} pu \rangle = 0, \tag{3}$$

where $\rho, u = (u^1, u^2, u^3), p, \gamma$ are mass density, particle velocity, gas pressure, ratio of the specific heats respectively, $\langle , \rangle$ represents scalar product and

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$$

is gradient in R^3.

Consider a piecewise smooth solution of these equations suffering a discontinuity of first kind on a smooth three dimensional surface Ω in R^4. Let Ω be represented by $s(x_1, x_2, x_3, t) = 0$. We choose the function s so that $|\nabla s|_\Omega = 1$. At any fixed time t, $s(x, t) = 0$ represents a smooth two-dimensional surface Ω_t in (x_1, x_2, x_3) space. Ω_t is then a shock surface and we call Ω *shock manifold*. The piecewise smooth solution of (1)–(3) can be represented in the form

$$\begin{bmatrix} \rho(x, t) \\ u(x, t) \\ p(x, t) \end{bmatrix} = \begin{bmatrix} \rho_0(x, t) \\ u_0(x, t) \\ p_0(x, t) \end{bmatrix} + H(s(x, t)) \begin{bmatrix} \rho_1(x, t) \\ u_1(x, t) \\ p_1(x, t) \end{bmatrix}, \tag{4}$$

where $\rho_i, u_i, p_i (i = 0, 1)$ are $C^\infty(R^4)$ functions and $H(\tau)$ is the Heaviside unit function defined by

$$H(\tau) = \begin{matrix} 1, & \tau > 0 \\ 0, & \tau \leqslant 0 \end{matrix}. \tag{5}$$

Assuming that (ρ_0, u_0, p_0) represents the known state ahead of the shock (the domain where $s < 0$) the problem is to find the successive positions of Ω_t and the state $(\rho_0+\rho_1, u_0+u_1, p_0+p_1)$ behind Ω_t satisfying some initial and boundary conditions.

Let R_Ω be the set of functions g of the form $g(x, t) = g_0(x, t) + H(s)g_1(x, t)$, where $g_0(x,t)$ and $g_1(x,t) \in C^\infty(R^4)$. R_Ω is closed with respect to the operations of addition, subtraction and multiplication. Therefore, all the combinations of ρ, u, p which appear under differentiation operators in (1)–(3) are also elements of R_Ω.

To a piecewise smooth function $g \in R_\Omega$, we associate a set of functions defined on the surface of discontinuity Ω by

$$g_{ij}|_\Omega = (\nabla s, \nabla)^j g_i|_\Omega, \quad i = 0, 1; j = 0, 1, 2, \ldots \tag{6}$$

These functions can be interpreted as successive derivatives of $g_i (i = 0, 1)$ along the normal to the shock front Ω_t at any fixed time t. We extend any function $g_{ij}|_\Omega$ defined on Ω in a smooth manner to the whole of R^4 so that the extended function g_{ij} is constant along the trajectories of the field $(\nabla s, \nabla)$ i.e., $\langle \nabla s, \nabla_{tx} \rangle g_{ij}|_\Omega = 0$. Then at any instant t, the extended function g_{ij} is constant along a normal to Ω_t. We note that ρ_{00} (or ρ_{10}) need not be equal to ρ_0 (or ρ_1) in R^4 except on Ω.

Let $\nabla_{tx} = (\partial/\partial t, \nabla)$ be the four-dimensional gradient in R^4. ∇ is gradient in (x_1, x_2, x_3)-space. However, whenever a scalar product of ∇ with an element of R^4 is taken, it is understood that

$$\nabla = \left(0, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right).$$

Then for $g \in R_\Omega$ and $e = (e^1, e^2, e^3, e^4) \in R^4$

$$\langle e - \nabla s \langle \nabla s, e \rangle, \nabla_{tx} \rangle g_{ij}|_\Omega = \langle e, \nabla_{tx} \rangle g_{ij}|_\Omega, \tag{7}$$

since $\langle \nabla s, \nabla_{tx} \rangle g_{ij}|_\Omega = \langle \nabla s, \nabla \rangle g_{ij}|_\Omega = 0$.

Let the class of generalised functions which are the results of differentiating functions of class R_Ω along some direction $e = (e^1, e^2, e^3, e^4)$ be denoted by $D_{e,\Omega}$. Let $g \in R_\Omega$ and $h(x, t) = \langle e, \nabla_{tx} \rangle g(x, t)$. Then $h \in D_{e,\Omega}$. To h we assign generalised set of functions $h_{ij}|_\Omega$, $i = 0, 1; j = -1, 0, 1, \ldots$, defined on Ω, by

$$\begin{aligned} h_{1,-1}(x, t)|_\Omega &= \{g_{10} \langle e, \nabla_{tx} \rangle s(x, t)\}|_\Omega, \\ h_{ij}(x, t)|_\Omega &= \{g_{i,j+1} \langle e, \nabla_{tx} \rangle s(x, t)\}|_\Omega \\ &\quad + \{\langle e - \nabla s \langle e, \nabla s \rangle, \nabla_{tx} \rangle g_{ij}\}|_\Omega, \\ &\quad i = 0, 1; j = 0, 1, \ldots \end{aligned} \tag{8}$$

Consider the class of generalised functions D_Ω representable as a finite sum of functions belonging to various sets $D_{e,\Omega}, e \in R^4$. Then if $h(x, t) \in D_\Omega$, there exists an $m > 0$, vectors $e_1, e_2, \ldots, e_m$ and a set of functions $h^1 \in D_{e_1,\Omega}, \ldots, h^m \in D_{e_m,\Omega}$ such that

$$h(x, t) = \sum_{i=1}^{m} h^i(x, t). \tag{9}$$

To the generalised function $h(x, t) \in D_\Omega$ we assign the generalised set $h_{ij}(x, t)$; $i = 0, 1$, $j = -1, 0, 1, 2, \ldots$ in the following manner

$$h_{ij}(x, t) = \sum_{k=1}^{m} h_{ij}^k(x, t), \tag{10}$$

where h_{ij}^k is the sequence associated with $h^i \in D_{e_{i,\Omega}}$.

We notice that for a solution of the form (4) the left side of the conservation laws (1)–(3) are actually the elements $h \in D_\Omega$. We next state the lemma which plays an important role in Maslov's method.

LEMMA. Let $g^1, \ldots, g^m$ be piecewise-smooth functions belonging to the class R_Ω of functions suffering discontinuity on the same smooth surface Ω and let $e^1, \ldots, e^m$ be a collection of vectors in R^4, if

$$\sum_{i=1}^{m} \langle e^i, \nabla_{tx} g^i \rangle = 0$$

then all terms of the series $\{h_{ij}(x,t)\}$, $i = 0, 1; j = -1, 0, 1, 2, \ldots$ defined by equations (10) vanish on the surface Ω.

For a proof of the lemma, reference may be made to Maslov's paper.

3. The SME and the Rankine-Hugoniot conditions

Using Maslov's lemma to the solution (4) of the system of conservation laws (1)–(3), we get the following expression for the column matrix $h_{1,-1}$

$$h_{1,-1}(x,t) = \begin{bmatrix} \rho_{10}\dfrac{\partial s}{\partial t} + \langle \nabla s, (\rho_{00}+\rho_{10})(u_{00}+u_{10}) - \rho_{00}u_{00} \rangle \\ \{(\rho_{00}+\rho_{10})(u_{00}^i+u_{10}^i) - \rho_{00}u_{00}^i\}\dfrac{\partial s}{\partial t} \\ +(\rho_{00}+\rho_{10})(u_{00}^i+u_{10}^i)\langle \nabla s, u_{00}+u_{10} \rangle \\ -\rho_{00}u_{00}^i \langle \nabla s, u_{00} \rangle + p_{10}\dfrac{\partial s}{\partial x_i} \\ \left[\frac{1}{2}(\rho_{00}+\rho_{10})\langle u_{00}+u_{10}, u_{00}+u_{10} \rangle\right. \\ \left. -\frac{1}{2}\rho_{00}\langle u_{00}, u_{00} \rangle\right]\dfrac{\partial s}{\partial t} + \dfrac{1}{(\gamma-1)}p_{10}\dfrac{\partial s}{\partial t} \\ +\frac{1}{2}(\rho_{00}+\rho_{10})\langle u_{00}+u_{10}, u_{00}+u_{10} \rangle \langle \nabla s, u_{00}+u_{10} \rangle \\ -\frac{1}{2}\rho_{00}\langle u_{00}, u_{00} \rangle \langle \nabla s, u_{00} \rangle \\ +\dfrac{\gamma}{(\gamma-1)}\{(p_{00}+p_{10})\langle \nabla s, u_{00}+u_{10} \rangle - p_{00}\langle \nabla s, u_{00} \rangle\} \end{bmatrix} = 0. \tag{11}$$

These are the usual jump conditions across a shock surface. Following the procedure followed by Prasad [3], we deduce the shock manifold equation (SME) in the form

$$\frac{\partial s}{\partial t} + \langle u_{00}, \nabla s \rangle + C|\nabla s| = 0, \tag{12}$$

where

$$C^2 = \frac{\rho_{00}+\rho_{10}}{\rho_{00}}\frac{p_{10}}{\rho_{10}} = \frac{(1+\mu)}{\mu\rho_0}p_{10},\ \mu = \frac{\rho_1}{\rho_0}. \tag{13}$$

The jump conditions across Ω are given by

$$u_{10}|_\Omega = -\xi(V_{10}p_{10})^{1/2}|_\Omega;\ \left.\frac{p_{00}+p_{10}}{p_{00}}\right|_\Omega = \left.\frac{\dfrac{(\gamma+1)}{(\gamma-1)}(1+\mu)-1}{\dfrac{(\gamma+1)}{(\gamma-1)}-(1+\mu)}\right|_\Omega \tag{14}$$

where $\xi = \dfrac{\nabla s}{|\nabla s|}$, $V_{10} = \dfrac{1}{\rho_{00}} - \dfrac{1}{\rho_{00}+\rho_{10}}$

$$C|_\Omega = -\left[2\left(\frac{\gamma p_0}{\rho_0}\right)\frac{(1+\mu)}{\{2+\mu(1-\gamma)\}}\right]^{1/2} \tag{15}$$

and

$$\mu = \rho_1/\rho_0 \tag{16}$$

represents the excess density ratio across the shock. a_0 is the local sound speed in the regions ahead of the shock.

The characteristics of the SME define the shock rays and are given by (we write here this result for a particular case of two spatial variables since we shall directly use it in the next section, x, y are spatial coordinates and Θ is the angle which the normal to the shock front makes with the x-axis)

$$\frac{dx_1}{d\tau} = u_{00}^1 + C\cos\Theta,\ \frac{dx_2}{d\tau} = u_{00}^2 + C\sin\Theta, \tag{17}$$

and

$$\frac{d\Theta}{d\tau} = \left(\sin\Theta\frac{\partial C}{\partial x_1} - \cos\Theta\frac{\partial C}{\partial x_2}\right) = \frac{\partial C}{\partial \eta}, \tag{18}$$

where $\partial/\partial\eta$ is tangential derivative along the shock curve Ω_t and $d/d\tau$ represents the operator for the temporal derivative of a quantity along a shock ray.

4. The second compatibility condition on the shock manifold

Maslov gives a method for the derivation of an infinite set of compatibility conditions on the shock manifold. The first group of such relations in this set is the Rankine-Hugoniot condition (14)–(15). The second group, namely the jump conditions on the first derivatives, consists of an expression for the rate of change of $\mu = \rho_1/\rho_0$ along a shock ray in terms of a tangential derivative of Θ in Ω_t and a normal derivative of μ and also some algebraic relations amongst the normal derivatives of the dependent variables. Our aim in this section is to derive one of the compatibility conditions in this group—namely the above mentioned expression for the rate of change of μ along the shock ray. For the discussion in the next section, we need only this part of the compatibility condition.

From Maslov's lemma for a solution (4) of (1) to (3) it follows that all the components

of the vector h_{10} vanish on Ω. This gives

$$\{\rho_{11} K_l + (\rho_{00}+\rho_{10}) \langle \nabla s, u_{11} \rangle + z\}|_\Omega = 0, \tag{19}$$

$$\rho_{11} K_l (u^i_{00}+u^i_{10}) + (\rho_{00}+\rho_{10})(u^i_{00}+u^i_{10}) \langle \nabla s, u_{11} \rangle + (\rho_{00}+\rho_{10}) u_{11} + p_{11} \frac{\partial s}{\partial x_i} + \omega^i|_\Omega = 0, \tag{20}$$

and

$$\tfrac{1}{2}\rho_{11} \langle u_{00}+u_{10}, u_{00}+u_{10} \rangle K_l + (\rho_{00}+\rho_{10}) \langle u_{00}+u_{10}, u_{11} \rangle K_l + \tfrac{1}{2}(\rho_{00}+\rho_{10}) \langle u_{00}+u_{10}, u_{00}+u_{10} \rangle \langle \nabla s, u_{11} \rangle + \frac{1}{(\gamma-1)} p_{11} \frac{\partial s}{\partial t} + \frac{\gamma}{(\gamma-1)} [(p_{00}+p_{10}) \langle \nabla s, u_{11} \rangle + p_{11} \langle \nabla s, u_{00}+u_{10} \rangle] + M|_\Omega = 0, \tag{21}$$

where

$$K_l = \frac{\partial s}{\partial t} + \langle \nabla s, u_{00}+u_{10} \rangle, \tag{22}$$

$$z = \frac{\partial \rho_{10}}{\partial t} + (\rho_{00}+\rho_{10}) \langle \nabla s, u_{01} \rangle + \rho_{01} \langle \nabla s, u_{00}+u_{10} \rangle - \langle \nabla s, \rho_{00} u_{01} + \rho_{01} u_{00} \rangle + \langle \nabla, (\rho_{00}+\rho_{10})(u_{00}+u_{10}) - \rho_{00} u_{00} \rangle, \tag{23}$$

$$\omega^i = \frac{\partial s}{\partial t} [(\rho_{00}+\rho_{10}) u^i_{01} + \rho_{01}(u^i_{00}+u^i_{10}) - \rho_{00} u^i_{01} - \rho_{01} u^i_{00}] + \frac{\partial}{\partial t} [(\rho_{00}+\rho_{10})(u^i_{00}+u^i_{10}) - \rho_{00} u^i_{00}] + (\rho_{00}+\rho_{10})(u^i_{00}+u^i_{10}) \langle \nabla s, u_{01} \rangle + (\rho_{00}+\rho_{10}) u^i_{01} \langle \nabla s, u_{00}+u_{10} \rangle + \rho_{01}(u^i_{00}+u^i_{10}) \langle \nabla s, u_{00}+u_{10} \rangle - \langle \nabla s, \rho_{00} u^i_{00} u_{01} + \rho_{00} u^i_{01} u_{00} + \rho_{01} u^i_{00} u_{00} \rangle + \langle u_{00}+u_{10}, \nabla \rangle (\rho_{00}+\rho_{10})(u^i_{00}+u^i_{10}) + (\rho_{00}+\rho_{10})(u^i_{00}+u^i_{10}) \langle \nabla, u_{00}+u_{10} \rangle - \rho_{00} u^i_{00} \langle \nabla, u_{00} \rangle - \langle u_{00}, \nabla \rangle \rho_{00} u^i_{00} + \frac{\partial p_{10}}{\partial x_i}, \tag{24}$$

$$M = \frac{\partial s}{\partial t} [\tfrac{1}{2}\rho_{01} \langle u_{00}+u_{10}, u_{00}+u_{10} \rangle + (\rho_{00}+\rho_{10}) \langle u_{00}+u_{10}, u_{01} \rangle - \tfrac{1}{2}\rho_{01} \langle u_{00}, u_{00} \rangle - \rho_{00} \langle u_{00}, u_{01} \rangle] + \frac{\partial}{\partial t} [\tfrac{1}{2}(\rho_{00}+\rho_{10}) \langle u_{00}+u_{10}, u_{00}+u_{10} \rangle \langle \nabla s, u_{00}+u_{10} \rangle - \tfrac{1}{2}\rho_{00} \langle u_{00}, u_{00} \rangle] + \frac{1}{(\gamma-1)} \frac{\partial p_{10}}{\partial t} + \tfrac{1}{2}\rho_{01} \langle u_{00}+u_{10}, u_{00}+u_{10} \rangle$$

$$\langle \nabla s,\, u_{00}+u_{10}\rangle + \tfrac{1}{2}(\rho_{00}+\rho_{10})\langle u_{00}+u_{10},\, u_{00}+u_{10}\rangle \langle \nabla s,\, u_{01}\rangle$$
$$+(\rho_{00}+\rho_{10})\langle u_{00}+u_{10}, u_{01}\rangle \langle \nabla s,\, u_{00}+u_{10}\rangle$$
$$-\tfrac{1}{2}\rho_{00}\langle u_{00},\, u_{00}\rangle \langle \nabla s,\, u_{01}\rangle - \tfrac{1}{2}\rho_{01}\langle u_{00},\, u_{00}\rangle \langle \nabla s,\, u_{00}\rangle$$
$$-\rho_{00}\langle u_{00},\, u_{01}\rangle \langle \nabla s,\, u_{00}\rangle + \tfrac{1}{2}(\rho_{00}+\rho_{10})\langle u_{00}+u_{10},\, u_{00}+u_{10}\rangle$$
$$\langle \nabla,\, u_{00}+u_{10}\rangle + \langle u_{00}+u_{10},\, \nabla\rangle\{\tfrac{1}{2}(\rho_{00}+\rho_{10})$$
$$\cdot\langle u_{00}+u_{10},\, u_{00}+u_{10}\rangle\} - \tfrac{1}{2}\rho_{00}\langle u_{00},\, u_{00}\rangle\langle \nabla,\, u_{00}\rangle$$
$$-\langle u_{00},\, \nabla\rangle\tfrac{1}{2}\rho_{00}\langle u_{00},\, u_{00}\rangle + \frac{\gamma}{(\gamma-1)}[p_{10}\langle \nabla s,\, u_{01}\rangle$$
$$+p_{01}\langle \nabla s,\, u_{10}\rangle + \langle \nabla,\, (p_{00}+p_{10})(u_{00}+u_{10}) - p_{00}u_{00}\rangle]. \tag{25}$$

Multiplying (19) by $(u_{00}+u_{10})$ and subtracting from (20) we get

$$(\rho_{00}+\rho_{10})\, u_{11}\, K_i + p_{11}\,\nabla s + \omega - (u_{00}+u_{10})z = 0. \tag{26}$$

Multiplying (19) by $\frac{1}{2}\langle u_{00}+u_{10},\, u_{00}+u_{10}\rangle$ and subtracting from (21) we get

$$(\rho_{00}+\rho_{10})\langle u_{00}+u_{10},\, u_{11}\rangle K_l + \frac{1}{(\gamma-1)}p_{11}\frac{\partial s}{\partial t}$$
$$+\frac{\gamma}{(\gamma-1)}[(p_{00}+p_{10})\langle \nabla s,\, u_{11}\rangle + p_{11}\langle \nabla s,\, u_{00}+u_{10}\rangle]$$
$$+M - \tfrac{1}{2}\langle u_{00}+u_{10},\, u_{00}+u_{10}\rangle z = 0. \tag{27}$$

Multiplying (26) by $(u_{00}+u_{10})$ and subtracting from (27) we get

$$\frac{1}{(\gamma-1)}p_{11}\,K_l + \frac{\gamma}{(\gamma-1)}(p_{00}+p_{10})\langle \nabla s,\, u_{11}\rangle$$
$$+M - \tfrac{1}{2}\langle u_{00}+u_{10},\, u_{00}+u_{10}\rangle z - \langle \omega,\, u_{00}+u_{10}\rangle$$
$$+\langle u_{00}+u_{10},\, u_{00}+u_{10}\rangle z = 0. \tag{28}$$

Multiplying (26) by $\dfrac{1}{(\gamma-1)}K_l\nabla s$ and subtracting from (28) we get

$$\alpha(\rho_{00}+\rho_{10})\langle \nabla s,\, u_{11}\rangle + M - \tfrac{1}{2}\langle u_{00}+u_{10},\, u_{00}+u_{10}\rangle z$$
$$-\langle \omega,\, u_{00}+u_{10}\rangle + \langle u_{00}+u_{10},\, u_{00}+u_{10}\rangle z - \frac{1}{(\gamma-1)}K_l\langle \nabla s,\, \omega$$
$$-(u_{00}+u_{10})z\rangle = 0 \tag{29}$$

where α is defined by (31).

Multiplying (19) by α and subtracting from (29) we get

$$M - \tfrac{1}{2}\langle u_{00}+u_{10},\, u_{00}+u_{10}\rangle z - \langle \omega,\, u_{00}+u_{10}\rangle$$
$$+z\langle u_{00}+u_{10},\, u_{00}+u_{10}\rangle - \frac{1}{(\gamma-1)}K_l\langle \omega-(u_{00}+u_{10})z,\, \nabla s\rangle$$
$$-\alpha z - \alpha\rho_{11}K_l = 0, \tag{30}$$

where

$$\alpha = \left\{ \frac{\gamma}{(\gamma-1)} \frac{p_{00}+p_{10}}{\rho_{00}+\rho_{10}} - \frac{K_I^2}{\gamma-1} \right\} = a_0^2 \frac{\mu(1+\gamma)}{(\gamma-1)(1+\mu)\{2+\mu(1-\gamma)\}}. \tag{31}$$

The operator $\mathrm{d}/\mathrm{d}\tau$ in (17) and (18) gives rate of change along a shock ray i.e., it represents an inner derivative in the shock manifold $\Omega: s(x, t) = 0$. Therefore, for a function $g_{ij}|_\Omega$ defined initially on Ω and extended as g_{ij} in R^4 so that it is constant on the trajectories of the field $\langle \nabla s, \nabla \rangle$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g_{ij}|_\Omega$$

can be obtained from $g_{ij}|_\Omega$ alone. From (14) it follows that $u_{10}|_\Omega$ is parallel to the vector $\nabla s|_\Omega$, hence $\langle u_{10}, \nabla \rangle g_{ij}|_\Omega = 0$. Therefore,

$$\left\{ \frac{\partial}{\partial t} + \langle u_{00}, \nabla \rangle \right\} g_{ij}|_\Omega = \left\{ \frac{\partial}{\partial t} + \langle u_{00}+u_{10}, \nabla \rangle \right\} g_{ij}|_\Omega$$

$$= \left\{ \frac{\partial}{\partial t} + \langle u_{00}+C\xi, \nabla \rangle \right\} g_{ij}|_\Omega, \ \xi = \frac{\nabla s}{|\nabla s|}. \tag{32}$$

Using these results, we find the following expressions for $M - \frac{1}{2}\langle u_{00}+u_{10}, u_{00}+u_{10} \rangle z$, $\langle \omega, u_{00}+u_{10} \rangle$, and $w - (u_{00}+u_{10})z$ on Ω:

$$M - \tfrac{1}{2}\langle u_{00}+u_{10}, u_{00}+u_{10} \rangle z|_\Omega =$$

$$-C[\rho_{01}\langle u_{10}, u_{10} \rangle + \tfrac{1}{2}\rho_{01}\langle u_{10}, u_{10} \rangle + \rho_{10}\langle u_{00}+u_{10}, u_{01} \rangle$$

$$+ \rho_{00}\langle u_{10}, u_{01} \rangle] + \frac{\mathrm{d}}{\mathrm{d}\tau}[\tfrac{1}{2}(\rho_{00}+\rho_{10})\langle u_{00}+u_{10}, u_{00}+u_{10} \rangle$$

$$-\tfrac{1}{2}\rho_{00}\langle u_{00}, u_{00} \rangle] - \tfrac{1}{2}\langle u_{00}+u_{10}, u_{00}+u_{10} \rangle \frac{\mathrm{d}(\rho_{00}+\rho_{10})}{\mathrm{d}\tau}$$

$$+ \tfrac{1}{2}\langle u_{00}+u_{10}, u_{00}+u_{10} \rangle \frac{\mathrm{d}\rho_{00}}{\mathrm{d}\tau} + \frac{1}{(\gamma-1)}\frac{\mathrm{d}p_{10}}{\mathrm{d}\tau}$$

$$+ \left[(\rho_{00}+\rho_{10})\langle u_{00}+u_{10}, u_{01} \rangle + \frac{\gamma p_{01}}{(\gamma-1)} \right] \langle \nabla s, u_{10} \rangle$$

$$+ \left[\rho_{00}\langle u_{00}, u_{10} \rangle + \tfrac{1}{2}\rho_{00}\langle u_{10}, u_{10} \rangle + \frac{\gamma p_{10}}{(\gamma-1)} \right] \langle \nabla s, u_{01} \rangle$$

$$-\tfrac{1}{2}\rho_{00}\langle u_{00}, u_{00} \rangle \langle \nabla, u_{00} \rangle + \tfrac{1}{2}\langle u_{00}+u_{10}, u_{00}+u_{10} \rangle \rho_{00}\langle \nabla, u_{00} \rangle$$

$$+ \frac{\gamma}{(\gamma-1)}[(p_{00}+p_{10})\langle \nabla, u_{00}+u_{10} \rangle + \langle u_{00}+u_{10}, \nabla \rangle p_{10}], \tag{33}$$

$$\langle \omega, u_{00}+u_{10} \rangle =$$

$$\langle u_{00}+u_{10}, \frac{\mathrm{d}}{\mathrm{d}\tau}\langle (\rho_{00}+\rho_{10})(u_{00}+u_{10}) - \rho_{00}u_{00} \} \rangle$$

$$+(\rho_{00}+\rho_{10})\langle u_{00}+u_{10},\, u_{00}+u_{10}\rangle\langle \nabla s,\, u_{01}\rangle$$
$$+\rho_{01}\langle u_{00}+u_{10},\, u_{00}+u_{10}\rangle\langle \nabla s,\, u_{10}\rangle-\rho_{00}\langle u_{00},\, u_{00}+u_{10}\rangle\langle \nabla s,\, u_{01}\rangle$$
$$+(\rho_{00}+\rho_{10})\langle u_{00}+u_{10},\, u_{01}\rangle\langle \nabla s,\, u_{10}\rangle$$
$$-C[\rho_{10}\langle u_{00}+u_{10},\, u_{01}\rangle+\rho_{01}\langle u_{00}+u_{10},\, u_{10}\rangle]$$
$$+(\rho_{00}+\rho_{10})\langle u_{00}+u_{10},\, u_{00}+u_{10}\rangle\langle \nabla,\, u_{00}+u_{10}\rangle$$
$$-\rho_{00}\langle u_{00}+u_{10},\, u_{00}\rangle\langle \nabla,\, u_{00}\rangle+\langle u_{00}+u_{10},\, \nabla\rangle p_{10}, \tag{34}$$

and

$$w-(u_{00}+u_{10})z=$$
$$-C[\rho_{10}u_{01}+\rho_{01}u_{10}]+\frac{\mathrm{d}}{\mathrm{d}\tau}[(\rho_{00}+\rho_{10})(u_{00}+u_{10})-\rho_{00}u_{00}]$$
$$-(u_{00}+u_{10})\frac{\mathrm{d}\rho_{10}}{\mathrm{d}\tau}+(\rho_{00}+\rho_{10})u_{01}\langle \nabla s,\, u_{10}\rangle$$
$$+\rho_{00}u_{10}\langle \nabla s,\, u_{01}\rangle+\rho_{00}u_{10}\langle \nabla,\, u_{00}\rangle+\nabla p_{10}. \tag{35}$$

Substituting these values in (30) we get

$$\frac{1}{(\gamma-1)}\frac{\mathrm{d}p_{10}}{\mathrm{d}\tau}+\frac{C\rho_0}{(\gamma-1)}\left\langle \nabla s,\, \frac{\mathrm{d}u_{10}}{\mathrm{d}\tau}\right\rangle+\frac{C\rho_0\mu}{(\gamma-1)(1+\mu)}\left\langle \nabla s,\, \frac{\mathrm{d}u_{00}}{\mathrm{d}\tau}\right\rangle$$
$$+\frac{C}{(\gamma-1)(1+\mu)}\langle \nabla s,\, u_{10}\rangle\frac{\mathrm{d}\rho_{00}}{\mathrm{d}\tau}+\rho_{00}\left\langle u_{10},\, \frac{\mathrm{d}u_{00}}{\mathrm{d}\tau}\right\rangle$$
$$-\tfrac{1}{2}\langle u_{10},\, u_{10}\rangle\frac{\mathrm{d}\rho_{00}}{\mathrm{d}\tau}-\alpha\frac{\mathrm{d}\rho_{10}}{\mathrm{d}\tau}+\rho_{01}\left[\tfrac{1}{2}\frac{C^3\mu}{(1+\mu)^2}-\alpha\langle \nabla s,\, u_{10}\rangle\right.$$
$$\left.-\frac{C^3\mu}{(\gamma-1)(1+\mu)^2}\right]+\langle \nabla s,\, u_{01}\rangle\left[\frac{-C^2\rho_0\mu}{(1+\mu)}-\tfrac{1}{2}\rho_0\frac{C^2\mu^2}{(1+\mu)^2}\right.$$
$$+\frac{\gamma c^2\mu\rho_0}{(\gamma-1)(1+\mu)}-\frac{c^2\mu\rho_0}{(\gamma-1)(1+\mu)}-\alpha\rho_{10}+\frac{c^2\mu\rho_0}{(\gamma-1)(1+\mu)}$$
$$\left.+\frac{C^2\mu\rho_0}{(\gamma-1)(1+\mu)^2}\right]+\frac{\gamma p_{01}}{(\gamma-1)}\langle \nabla s,\, u_{10}\rangle$$
$$+\left[\frac{\gamma}{(\gamma-1)}(p_{00}+p_{10})-\alpha(\rho_{00}+\rho_{10})\right]\langle \nabla,\, u_{10}\rangle$$
$$+\left[\frac{\gamma}{(\gamma-1)}p_{10}+\frac{C^2\mu\rho_0}{(\gamma-1)(1+\mu)^2}-\alpha\rho_{10}-\tfrac{1}{2}\langle u_{10},\, u_{10}\rangle\right]\langle \nabla,\, u_{00}\rangle. \tag{36}$$

Since the operator $\mathrm{d}/\mathrm{d}\tau$ gives an interior derivative in Ω, we note that in the above we can use

$$\mathrm{d}\rho_{00}/\mathrm{d}\tau=\mathrm{d}\rho_0/\mathrm{d}\tau,\quad \frac{\mathrm{d}}{\mathrm{d}\tau}(\rho_{00}u_{00})=\frac{\mathrm{d}(\rho_0u_0)}{\mathrm{d}\tau}$$

etc. The set $\{\rho_0(x,t), u_0(x,t), p_0(x,t)\}$ is a genuine solution of equations (1) to (3), hence

$$\left.\frac{d\rho_{00}}{d\tau}\right|_\Omega = \left.\frac{d\rho_0}{d\tau}\right|_\Omega = \left.\left(\frac{\partial\rho_0}{\partial t} + \langle u_0, \nabla\rangle\rho_0 + C\langle \nabla s, \nabla\rangle\rho_0\right)\right|_\Omega$$
$$= -\rho_0\langle\nabla, u_0\rangle + C\rho_{01}|_\Omega. \tag{37}$$

Similarly, we get

$$\left.\frac{du_{00}}{d\tau}\right|_\Omega = \frac{1}{\rho_0}(C\rho_0 u_{01} - \nabla p_0), \tag{38}$$

and

$$\rho_{01} = \frac{1}{C}\left(\rho_0\langle\nabla, u_0\rangle + \frac{d\rho_{00}}{d\tau}\right). \tag{39}$$

We also note

$$\left\langle \xi, \frac{d\xi}{d\tau}\right\rangle = 0, \langle\xi, L\rangle|_\Omega = 0,$$
$$\langle\nabla, u_0\rangle|_\Omega = \langle\nabla, u_{00}\rangle|_\Omega + \langle\nabla s, u_{01}\rangle|_\Omega, \tag{40}$$

where

$$L = \nabla - \xi\langle\xi, \nabla\rangle. \tag{41}$$

In terms of the rate of change of the excess density ratio, $\mu = (\rho_1/\rho_0)$ across the shock, eliminating p_{10} and u_{10} from (14) and (15) respectively and using (33) to (36) we write (32) in the form

$$\left\{\frac{2C^2\rho_0}{(\gamma-1)(1+\mu)^2} + \frac{2C\mu\rho_0}{(\gamma-1)(1+\mu)}\frac{\partial c}{\partial\mu} - \alpha\rho_0\right\}\frac{d\mu}{d\tau}$$
$$+\left\{\frac{C^2\mu}{(\gamma-1)(1+\mu)} + \frac{3C\mu\rho_0}{(\gamma-1)(1+\mu)}\frac{\partial c}{\partial\rho_0} - \alpha\mu\frac{(2+\mu)}{(1+\mu)}\right\}\frac{d\rho_{00}}{d\tau}$$
$$+\frac{3C\mu\rho_0}{(\gamma-1)(1+\mu)}\frac{\partial C}{\partial p_0}\frac{dp_{00}}{d\tau} + \frac{C^2\mu\rho_0}{(\gamma-1)(1+\mu)^2}\langle L, \xi\rangle$$
$$+\frac{C^2\mu\rho_0}{(\gamma-1)(1+\mu)}\langle\nabla s, u_{01}\rangle + \left[\frac{\gamma}{(\gamma-1)}\frac{C^2\mu\rho_0}{(1+\mu)} -\right.$$
$$\left. -\frac{\alpha\mu\rho_0(2+\mu)}{(1+\mu)}\right]\langle\nabla, u_0\rangle + \left.\frac{\alpha C\rho_{11}}{(1+\mu)}\right|_\Omega = 0. \tag{42}$$

This is the required compatibility condition giving the rate of change of μ along a shock ray in terms of the known quantities in the region ahead of the shock, the interior derivatives of unit normal ξ of the shock front Ω_t and a normal derivative ρ_{11}.

5. Correct form of the shock dynamic equations in two-space dimensions

In the case of two space variables x_1, x_2, the shock Ω_t at time t will be represented by a curve in (x_1, x_2) plane. Let Θ be the angle which the normal to Ω_t makes with the x_1-axis.

Then $\xi = (\cos\Theta, \sin\Theta)$. Assuming the state ahead of the shock to be a constant state and at rest, i.e. $u_0^1 = 0$, $u_0^2 = 0$, and using $\langle L, \xi \rangle = \partial\Theta/\partial\eta$ we get from (42)

$$\frac{\partial\mu}{\partial s_1} = B_1 \frac{\partial\Theta}{\partial\eta} + B_2 \frac{\partial\mu}{\partial N}, \tag{43}$$

where ds_1 is an element of length described in time dt in (x_1, x_2)-plane while moving along a shock ray ($ds_1 = C\,dt$), $d\eta$ is an element of length along the shock curve Ω_t at any time and dN is an element of length in a direction normal to Ω_t:

$$\frac{\partial}{\partial s_1} = \frac{1}{C}\left(\frac{\partial}{\partial t} + (u_0^1 + C\cos\Theta)\frac{\partial}{\partial x_1} + (u_0^2 + C\sin\Theta)\frac{\partial}{\partial x_2}\right) \tag{44}$$

$$\frac{\partial}{\partial\eta} = \sin\Theta\frac{\partial}{\partial x_1} - \cos\Theta\frac{\partial}{\partial x_2}, \tag{45}$$

and

$$\frac{\partial}{\partial N} = \cos\Theta\frac{\partial}{\partial x_1} + \sin\Theta\frac{\partial}{\partial x_2}. \tag{46}$$

The coefficients B_1 and B_2 in (43) are given by

$$B_1 = -\frac{2\mu\{2+\mu(1-\gamma)\}}{8+6\mu-2\gamma\mu-\mu^2(1-\gamma^2)} \tag{47}$$

and

$$B_2 = -\frac{\mu(1+\gamma)\{2+\mu(1-\gamma)\}}{\{8+6\mu-2\gamma\mu-\mu^2(1-\gamma^2)\}(1+\mu)}. \tag{48}$$

Substituting in (18) the expression for C in terms of μ, we get

$$\frac{\partial\Theta}{\partial s_1} = -\frac{(\gamma+1)}{2(1+\mu)\{2+\mu(1-\gamma)\}}\frac{\partial\mu}{\partial\eta}. \tag{49}$$

Equations (43) and (49) could be used to calculate the successive positions of a shock front provided the term $\partial\mu/\partial N$ can be estimated in (43). This term can be approximately evaluated for a class of problems when the shock is weak [4]. However, since (43) and (49) are exact equations, it would be interesting to compare these with the corresponding equations of Whitham's shock dynamics [7, 8], which even today, is the only successful nonnumerical technique for the determination of the successive positions of a curved shock. We shall first give here Whitham's shock dynamics equations for shock motion in two space dimensions.

Assuming that the state ahead of the shock is at rest the successive positions of the shock and the corresponding family of shock rays form a set of orthogonal curvilinear coordinates. Let $\alpha = a_0 t$ = constant represent the family of curves giving the successive positions of the shock and β = constant represent the family of shock rays. The line elements along the two families of curves are $M\,d\alpha$ and $A\,d\beta$, where M is the shock Mach number and A is an appropriately defined area of an elementary shock ray tube. Considering the changes in the shock ray inclination Θ and the shock Mach number along the shock curve, we get the following two equations in Θ, M and A:

$$\partial\Theta/\partial\alpha = -(1/A)(\partial M/\partial\beta), \tag{50}$$

and

$$\partial A/\partial\alpha = M\,(\partial\Theta/\partial\beta). \tag{51}$$

To obtain a determinate system of equations, Whitham assumed that the flow in each ray tube can be treated as one-dimensional, and a functional relation between A and M can be obtained by using the CCW approximation, in the form $A = A_0 f(M)/f(M_0)$ where $f(M)$ is given by

$$f(M) = \exp\left\{-\int \frac{M\lambda(M)}{M^2-1}\,\mathrm{d}M\right\}$$

$$\lambda(M) = \left\{1+\frac{2}{(\gamma+1)}\frac{(1-d^2)}{d}\right\}\left\{1+2d+\frac{1}{M^2}\right\}$$

$$d^2 = \frac{(\gamma-1)M^2+2}{2\gamma M^2-(\gamma-1)}.$$

Once Θ and M are determined as functions of α, β, the position of the shock at any time $t = \alpha/a_0$ can be obtained by integrating the ray equations

$$\partial x/\partial\alpha = M\cos\Theta,\ \partial y/\partial\alpha = M\sin\Theta. \tag{52}$$

The operators $\partial/\partial\alpha$ and $\partial/\partial\beta$ give interior derivatives in the shock manifold Ω. In order to compare the compatibility conditions of the shock dynamics with those derived by Maslov's method, we write (50) and (51) in terms of the excess density ratio μ across the shock and in terms of spatial distances s and η related to α and β by $\mathrm{d}s_1 = M\,\mathrm{d}\alpha$ and $\mathrm{d}\eta = A\,\mathrm{d}\beta$ respectively. The results are

$$\partial\mu/\partial s_1 = B_3(\partial\Theta/\partial\eta) \tag{53}$$

and

$$\frac{\partial\Theta}{\partial s_1} = -\frac{(\gamma+1)}{2(1+\mu)\{2+\mu(1-\gamma)\}}\frac{\partial\mu}{\partial\eta} \tag{54}$$

where

$$B_3 = \frac{-2^{3/2}\{2+\mu(1-\gamma)\}(1+\mu)[2+(\gamma+1)\mu]^{1/2}}{\sqrt{2}\{4+\mu(3-\gamma)\}[2+(\gamma+1)\mu]^{1/2}+(\gamma+1)\mu\{4+\mu(3-\gamma)\}+8(1+\mu)}. \tag{55}$$

We note that (54) of shock dynamics is exactly the same as (49) derived by Maslov's method. However, the second term $B_2(\partial\mu/\partial N)$ of (43) is missing in (53) of shock dynamics. This term is of the same order as the first two terms unless the state behind the shock is almost a uniform state. The limiting form of (43) and (49) have been derived for a weak shock by Ramanathan *et al* [5] by another method. We note that, in this case, i.e. $\mu \ll 1$, both coefficients B_1 (of Maslov's method) and B_3 (of shock dynamics) can be approximated by the same value $-\frac{1}{2}\mu$ as derived in [5]. For a shock of arbitrary strength table 1 gives a comparison of the values of B_1 and B_3. Unlike the intuitive and approximate deduction of the shock dynamic equations, Maslov's theory gives the exact results by a mathematically elegant method. The authors believe that Maslov's theory will be of great value in the theory of multi-dimensional shock propagation.

Table 1. Comparison between Whitham's result and Maslov's result

μ	B_1	B_3
0	0.00	0.00
0.5	−0.18292683	−0.22842144
1	−0.26315789	−0.41225898
1.5	−0.28074866	−0.54515265
2	−0.6315796	−0.62393547
2.5	−0.22727273	−0.64814813
3	−0.18292683	−0.61873780
3.5	−0.13565892	−0.53731328
4	−0.088495586	−0.40575466
4.5	−0.043021033	−0.22601149
5	$0.12417634 \times 10^{-7}$	$0.74066669 \times 10^{-7}$

For $\mu = 5$, the table shows that B_3 is approximately six times that of B_1

6. Conservation form of the equations on a shock manifold and the jump relations for the shock-shock

In the last section we have derived two equations for the amplitude μ of the shock and for the angle Θ made by the normal to the shock front with the positive directions of the x_1-axis. As these equations are exact, it is natural and also interesting to analyse these equations further.

The SME (12) reduces to

$$\partial s/\partial t + C\,|\nabla s| = 0,$$

when the state ahead of the shock front is at rest.

For the subsequent part of the analysis we assume that s is given by

$$s(x_\alpha, t) = t + \phi(x_\alpha),$$

then

$$1 + C(\phi_{x_2}\phi_{x_2})^{1/2} = 0. \tag{56}$$

Let us make a coordinate transformation from (x_α, t) to (x'_α, s) given by

$$x'_\alpha = x_\alpha,\ s = s(x_\alpha, t) = t + \phi(x_\alpha). \tag{57}$$

In this coordinate system (56) becomes

$$1 + C(\phi_{x'_2}\phi_{x'_2})^{1/2} = 0. \tag{58}$$

The characteristic curves of (58) or shock rays are given by

$$\mathrm{d}x'_\alpha/\mathrm{d}\tau = C\phi_{x'_2}/(\phi_{x'_\beta}\phi_{x'_\beta})^{1/2} = CN_\alpha, \tag{59}$$

where $N_1 = \cos\Theta$, $N_2 = \sin\Theta$ are the components of the unit normal to the shock curve $\phi(x'_\alpha) = \text{constant}$.

The directional derivative operator along shock rays is

$$\mathrm{d}/\mathrm{d}\tau = CN_\alpha \frac{\partial}{\partial x'_\alpha} \tag{60}$$

From (54) we get

$$\frac{d\Theta}{d\tau} = \left(\sin\Theta\frac{\partial C}{\partial x_1'} - \cos\Theta\frac{\partial C}{\partial x_2'}\right), \tag{61}$$

which with the help of (60) can be written as

$$CN_\alpha\frac{\partial\Theta}{\partial x_\alpha'} - \left(\sin\Theta\frac{\partial C}{\partial x_1'} - \cos\Theta\frac{\partial C}{\partial x_2'}\right) = 0$$

or $\nabla'(\mathbf{N}'/C) = 0$ on Ω, (62)

where $\nabla' = \left(\frac{\partial}{\partial x_1'}, \frac{\partial}{\partial x_2'}\right)$ is the two-dimensional gradient in (x_α') coordinate system and $\mathbf{N}' = (\sin\Theta, -\cos\Theta)$ is the unit vector tangential to the shock curve. Equation (62) is a conservation form of one of our shock dynamic equations.

Let us integrate the expression on the left side of (62) over the region *ABCD* as shown in figure 1 where *AD*, *BC* are two positions of shock fronts, AB, CD are the shock rays. Then

$$\iint_{\text{area ABCD}} \nabla'\cdot(\mathbf{N}'/C) = 0 \text{ leads to}$$

$$\int_{(AB+BC+CD+DA)} \frac{\mathbf{N}'\cdot\nu}{C}dl = 0$$

ν being the exterior unit normal to the curve ABCD. On AD and BC $\nu\cdot\mathbf{N}' = 0$, on AB $\nu\cdot\mathbf{N}' = 1$ and on CD $\nu\cdot\mathbf{N}' = -1$. Hence we get

$$\int_{AB}\frac{dl}{C} = \int_{CD}\frac{dl}{C}.$$

Since C is the velocity of the shock, the above equation implies that the shock front at any time t can be obtained by moving a distance Ct along the rays starting from its initial position. Let us analyse (43) which can be written as

$$\frac{d\mu}{d\tau} - B_1\frac{\partial\Theta}{\partial\eta} = B_2\frac{\partial\mu}{\partial N}. \tag{63}$$

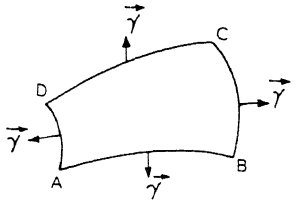

Figure 1. Elementary area whose boundaries are two positions of the shock front and two shock rays

The right side expression in (63) represents the derivative of μ in the direction of normal to the shock front and accounts for the effect of flow from behind the shock. The flow behind the shock front depends on initial conditions and boundary conditions which can be chosen arbitrarily. Therefore, by choosing initial and boundary conditions in a proper way the term $B_2 \dfrac{\partial \mu}{\partial N}$ can be made to assume any given value. Thus for the kinematics of a shock front this term should be treated as a known quantity.

Since $$\frac{\partial \Theta}{\partial \eta} = -\nabla \cdot \mathbf{N} = -\nabla' \cdot \mathbf{N}' \qquad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\tau} = CN_\alpha \frac{\partial}{\partial x'_\alpha}$$

equation (63) becomes

$$CN_\alpha \frac{\partial \mu}{\partial x'_\alpha} + B_1(\mu) \frac{\partial N_\alpha}{\partial x'_\alpha} = B_2(\mu) \frac{\partial \mu}{\partial N} \tag{64}$$

We shall show that there exists a function $f(\mu)$ so that when (64) is multiplied by it, it takes the form

$$\nabla' \cdot (K(\mu)\mathbf{N}) = F(\mu, x'_1, x'_2),$$

where $F(\mu, x'_1, x'_2)$ is a known function of μ, x_1, x_2 and $K(\mu)$ is a function of μ. Such a factor $f(\mu)$ has to satisfy the equations

$$C(\mu) f(\mu) = K'(\mu) \text{ and } B_1(\mu)\, f(\mu) = K(\mu),$$

which implies $(B_1 f)' = Cf$ or $B_1 \times f' = f(C - B'_1)$.

Thus $$f(\mu) = \left\{ \exp\left[\int \frac{(C - B'_1)}{B_1} \mathrm{d}\mu \right] \right\}$$

Hence $f(\mu)$ is determined upto a multiplicative constant. This constant can be taken to be one. Hence (64) can be written as

$$\nabla' \cdot (K(\mu)\mathbf{N}) = F(\mu). \tag{65}$$

Let us write explicitly the two conservation laws (62) and (65) on the shock manifold Ω.

$$\frac{\partial}{\partial x'_1}\left(\frac{\sin \Theta}{C}\right) + \frac{\partial}{\partial x'_2}\left(\frac{-\cos \Theta}{C}\right) = 0, \tag{66}$$

and

$$\frac{\partial}{\partial x'_1}(\mathrm{K} \cos \Theta) + \frac{\partial}{\partial x'_2}(K \sin \Theta) = F(\mu). \tag{67}$$

The smooth solutions of the quasilinear equations derived from (66) and (67) develop into discontinuities in μ and Θ which are shock-shocks. From (66) and (67) we derive the jump condition across a shock-shock.

If $U = \mathrm{d}x'_2/\mathrm{d}x'_1$ is the slope of the shock-shock curve in (x'_1, x'_2) plane then (66) and (67)

$$-U[K \cos \Theta] + [K \sin \Theta] = 0,$$

and

$$-U\left[\frac{\sin \Theta}{C}\right] + \left[\frac{-\cos \Theta}{C}\right] = 0,$$

where [ ] on any quantity represents its jump across the shock-shock.

or

$$U = \frac{(K_2 \sin\Theta_2 - K_1 \sin\Theta_1)}{(K_2 \cos\Theta_2 - K_1 \sin\Theta_1)} = \frac{(C_2 \cos\Theta_1 - C_1 \cos\Theta_2)}{(C_1 \sin\Theta_2 - C_2 \sin\Theta_1)}, \tag{68}$$

where $K_2 = K(\mu_2), K_1 = K(\mu_1), C_2 = C(\mu_2), C_1 = C(\mu_1)$ and subscripts 1 and 2 refer to the states on the two sides of shock-shock. Eliminating U from these two we get

$$\tan(\Theta_1 - \Theta_2) = \frac{(K_1^2 - K_2^2)^{1/2}(C_1^2 - C_2^2)^{1/2}}{(K_1 C_2 + K_2 C_1)}. \tag{69}$$

Whitham derived the jump relations across a shock-shock by considering the invariance of the distance traversed by the shock-shock. In contrast to this, we have derived these jump relations directly from the appropriate conservation laws which are also exact. We have given justification why the term $B_2(\partial\mu/\partial N)$ in (65) should not contribute to the jump relations.

References

[1] Courant R and Friedrichs K O 1948 *Supersonic flow and shock waves.*

[2] Maslov V P 1980 Propagation of shock waves in the isentropic nonviscous gas. *J. Sov. Maths.* **13,** 119–163

[3] Prasad P 1982 Kinematics of a multi-dimensional shock of arbitrary strength in an ideal gas. *Acta Mech.* **45** 163–176

[4] Ramanathan T M, Prasad P and Ravindran R 1984 On the propagation of a weak shock front: theory and application. *Acta Mech.* **51** 167–177

[5] Ramanathan T M, Prasad P and Ravindran R On the question of resolution of a hyperbolic caustic due to nonlinearity (sent for publication)

[6] Richtmyer R D and Morton K W 1967 *Difference methods for initial value problems* (New York: Interscience)

[7] Whitham G B 1957 A new approach to problems of shock dynamics Part I *J. Fluid Mech.* **2** 146–171

[8] Whitham G B 1959 A new approach to problems of shock dynamics, Part II, Three-dimensional problems. *J. Fluid Mech.* **5** 369–386

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Distance characterizations of finite lattices

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Abstract. Semimodular, modular and distributive finite lattices are characterized by means of convex sublattices and distance closed sets of the Hasse diagram graphs.

Keywords. Finite lattices; distance closed sets; convex sublattices; distributivity; modularity; semimodularity.

1. Introduction

A nonempty set D of a finite lattice L is distance closed, if for any $a, b \in D$ the set D contains every element on every shortest ab-path in the (undirected) Hasse diagram graph of L. One can show that every convex sublattice C of a finite semi-modular lattice L is a distance closed set of L. The converse does not hold. The purpose of this paper is to characterize different finite lattices by means of distance closed sets. As the definition of a distance closed set shows, the concept is of graph theoretic nature, and thus the characterizations of lattices here are graph theoretic ones that Grätzer asked for in his book [2, p. 53].

If A and B are two distance closed sets of a finite L, then $A \cap B$ is distance closed, too, provided that $A \cap B \neq \varnothing$. Thus, if $a, b \in L$, there is a least distance closed set containing a and b; that set will be denoted by $\langle a, b \rangle$.

According to Glivenko (see e.g. [1, II:9]), the distance equation $d(a, c) = d(a, b) + d(b, c)$ holds for a, b, c in any finite modular lattice if and only if $(a \wedge b) \vee (b \wedge c) = b = (a \vee b) \wedge (b \vee c)$.

2. Characterizations

At first we give a characterization for modular lattices.

THEOREM 1.
A finite lattice L is modular if and only if every distance closed subset D of L is a convex sublattice of L.

Proof. Let L first be modular, D a distance closed set of L, and $a, b \in D$. Now $(a \wedge (a \vee b)) \vee (b \wedge (a \vee b)) = a \vee b = (a \vee (a \vee b)) \wedge (b \vee (a \vee b))$, and because L is modular, this means by Glivenko that $d(a, b) = d(a, a \vee b) + d(a \vee b, b)$. Thus $a \vee b$ is on a shortest ab-path, whence $a \vee b \in D$. Dually we see that $a \wedge b \in D$. This proves that D

is a sublattice of L. Let now $a \geqslant c \geqslant b$, and $a, b \in D$. Then $(a \wedge c) \vee (c \wedge b) = c = (a \vee c) \wedge (b \vee c)$, whence $d(a,b) = d(a,c) + d(c,b)$. Thus $c \in D$ and D is convex. Accordingly, D is a convex sublattice of L.

Let conversely every distance closed set D of L be a convex sublattice. If L is not modular, then either the upper or the lower covering condition does not hold [2, IV:2:Cor. 3]. If the upper covering condition does not hold, there is a pair a, b of elements such that $a \neq b$, $a \wedge b \prec a, b$ and $a, b \not\prec a \vee b$. But then the least distance closed set $\langle a, b \rangle$ contains only the elements a, b and $a \wedge b$, and thus $\langle a, b \rangle$ is not a convex sublattice of L; a contradiction. The validity of the lower covering condition can be proved dually, and thus L is modular.

Next we try to get a distance characterization for distributive lattices and, on the other hand, for modular nondistributive lattices. Some new concepts are first needed. When $a, b \in L$, we define $SP(a,b) = \{x \mid x \in L$ and x is on a shortest ab-path$\} = SP^1(a,b)$. If A and B are two nonempty subsets of L, then $SP(A,B) = \{x \mid x \in L$ and x is on a shortest ab-path, $a \in A$ and $b \in B\} = SP^1(A,B)$. In general, $SP^{n+1}(a,b) = SP(SP^n(a,b), SP^n(a,b))$, and because L is finite, $SP^m(a,b) = SP^{m+1}(a,b) = SP^{m+2}(a,b) = \ldots$ for some finite integer $n = m$. Clearly $\langle a, b \rangle = SP^m(a,b)$. We say that a distance closed set D is of degree one, if $\langle a, b \rangle = SP(a,b)$ for all two elements $a, b \in D$, and generally, D is of degree k ($\geqslant 2$), if $\langle a, b \rangle = SP^t(a,b)$ with $t \leqslant k$ for all $a, b \in D$, and $\langle a, b \rangle = SP^k(a,b) \neq SP^{k-1}(a,b)$ for at least one pair $a, b \in D$.

If a finite lattice L is modular and nondistributive, it contains at least one nondistributive and modular sublattice M_3 (the diamond) of five elements. One can show that every M_3 of L is a convex sublattice of L. Now we are ready to prove

THEOREM 2.
A finite lattice L is distributive if and only if every distance closed set is of degree one and a convex sublattice of L.

Proof. According to the previous theorem we can concentrate on the degree of freedom only.

Let L be distributive, D a distance closed subset, and $a, b \in D$. Clearly $SP(a,b) \subset \langle a, b \rangle$. On the other hand, $\langle a, b \rangle$ is also a convex subalgebra, and because it is the least one containing a and b, it is the interval $[a \wedge b, a \vee b]$. We shall show that $[a \wedge b, a \vee b] \subset SP(a,b)$, which implies the desired equality $\langle a, b \rangle = SP(a,b)$. Now if $c \in [a \wedge b, a \vee b]$, then $(c \wedge a) \vee (c \wedge b) = c \wedge (a \vee b) = c = c \vee (a \wedge b) = (c \vee a) \wedge (c \vee b)$, whence $d(a,b) = d(a,c) + d(c,b)$ and $c \in SP(a,b)$. Thus $\langle a, b \rangle = SP(a,b)$ and D is of degree one.

Let, conversely, every distance closed set D of L be of degree one and a convex sublattice of L. According to Theorem 1, L is modular. If L is nondistributive, it contains M_3 as a convex sublattice. Let, in M_3, $e < a, b, c < i$. Now $\langle a, b \rangle = M_3$ and $c \notin SP(a,b)$, whence $\langle a, b \rangle \neq SP(a,b)$. This contradicts the condition of the theorem, and hence L is distributive.

The proof of the following theorem is obvious according to the latter part of the preceding proof, and hence we omit it.

THEOREM 3.
A finite lattice L is modular and nondistributive if and only if every distance closed set D of L is a convex subalgebra of L and at least one D is of degree two.

The last theorem of this paper gives a characterization for finite semimodular lattices. By combining it with its dual we obtain the characterization for modular lattices given in Theorem 1.

THEOREM 4.
A finite lattice L is semimodular if and only if every distance closed set D of L is a convex $\vee$-sublattice of L.

Proof. Let L be semimodular, D a distance closed set, and $a, b \in D$. Let one shortest ab-path consist of pieces of chains as follows: $a = c_0 \nearrow c_1 \searrow c_2 \nearrow c_3 \searrow \ldots \nearrow c_{p-1} \searrow c_p = b$. Because the path is the shortest ab-path with increasing and decreasing chains indicated by arrows, we have $a \vee c_2 = c_0 \vee c_2 = c_1$, $c_1 \wedge c_3 = c_2$, $c_2 \vee c_4 = c_3, c_3 \vee c_5 = c_6, \ldots c_{p-2} \vee b = c_{p-2} \vee c_p = c_{p-1}$. Let the chain from c_2 to c_1 contain the elements $c_2 = e_0 \prec e_1 \prec e_2 \prec \ldots \prec e_n = c_1$ and the chain from c_2 to c_3 the elements $c_2 = f_0 \prec f_1 \prec f_2 \prec \ldots \prec f_k = c_3$. In a semimodular lattice, if $s \wedge t \prec s, t$ then $s, t \prec s \vee t$. Now $e_0 \prec e_1, f_1$, and thus by the semi-modularity, $e_1, f_1 \prec e_1 \vee f_1$. Further $f_1 \prec f_2, e_1 \vee f_1$, and thus $f_2, e_1 \vee f_1 \prec e_1 \vee f_1 \vee f_2 = e_1 \vee f_2$. By repeating this process, we obtain that $e_1 \vee f_1 \vee f_2 \vee \ldots \vee f_k = e_1 \vee f_k \succ e_1 \vee f_{k-1}, f_k$. Accordingly, the chain from e_1 to $e_1 \vee f_k$ is of the same length as the chain from e_0 to f_k. By substituting now in the foregoing process e_0 by e_1, e_1 by e_2 and the chain $e_0, f_1, \ldots, f_k$ by $e_1, e_1 \vee f_1, \ldots, e_1 \vee f_k$, we see that $e_2 \prec e_2 \vee f_1 \prec e_2 \vee f_2 \prec \ldots \prec e_2 \vee f_k$. As above, $d(e_2, e_2 \vee f_k) = d(e_0, f_k)$. By continuing the process, we see that $e_n \prec e_n \vee f_1 \prec e_n \vee f_2 \prec \ldots \prec e_n \vee f_k$ and $d(e_n, e_n \vee f_k) = d(e_0, f_k)$. By the same way we see that $d(e_0, e_n) = d(f_k, e_n \vee f_k)$. But then there is a shortest ab-path also over $e_n \vee f_k = c_1 \vee c_3$. Thus one shortest ab-path is $a = c_0 \nearrow c_1 \vee c_3 \searrow c_4 \nearrow c_5 \searrow \ldots \searrow c_p = b$, where $c_1 \vee c_3 = a \vee c_4$, because the path is a shortest one. By applying the process above repeatedly, we see that one shortest ab-path goes over $c_1 \vee c_3 \vee c_5 \vee \ldots \vee c_p$, where $c_1 \vee c_3 \vee c_5 \vee \ldots \vee c_p = a \vee b$, because the path is a shortest one. Thus $a \vee b \in D$ in this case. The proof is analogous for all shortest ab-paths and thus $a \vee b \in D$ in any case. Hence D is a $\vee$-sublattice. Let $b \leqslant c \leqslant a$ and $a, b \in D$. In a finite semimodular lattice every ab-chain is of the same length, and thus $d(a, b) = d(a, c) + d(c, b)$. This implies that $c \in D$ and hence D is convex, too. This proves that every distance closed set D is a convex $\vee$-sublattice of L.

Let conversely every distance closed set be a convex $\vee$-sublattice of L. Assume that L is not semimodular, whence there are elements a and b such that $a \neq b$, $a \wedge b \prec a, b$ and $a, b \not\prec a \vee b$. Then $\{a \wedge b, a, b\}$ is a distance closed set in L without being a $\vee$-sublattice. This is a contradiction, and hence L is semimodular. This completes the proof.

References

[1] Birkhoff G 1973 *Lattice theory* (Rhode Island: American Math. Soc.) p. 42
[2] Grätzer G 1978 *General lattice theory* (Basel und Stuttgart: Birkhäuser)

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Closed set characterizations of finite lattices

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Abstract. Semimodularity and the Jordan-Hölder chain condition are characterized in a finite lattice L by means of special closed sets of L.

Keywords. Finite lattices; convex sublattices; closed sets; semimodularity; Jordan-Hölder chain condition.

1. Introduction and terminology

In the earlier paper [1] finite semimodular, modular and distributive lattices were characterized by means of distance closed sets. The purpose of this paper is to find new closed sets by means of which the semimodularity, the modularity and the Jordan-Hölder chain condition can be characterized in finite lattices.

Let L be a finite lattice. If $a, b \in L$, then $a \succ b$ means that $a > b$ and a covers b, i.e. if there is an element $c \in L$ with $a \geqslant c \geqslant b$, then either $a = c$ or $b = c$. A lattice L is semimodular, if for any $a, b \in L$ with $a, b \succ a \wedge b$ also $a \vee b \succ a, b$ holds. The dual semimodularity is defined dually. A lattice L is modular if and only if L is semimodular and dually semimodular. A finite lattice L satisfies the Jordan-Hölder chain condition if any two maximal chains are of the same length.

Let S be a nonempty subset of a finite lattice L. S is connected, if for any two elements t and q of S there is a sequence $t = s_1, s_2, \ldots, s_m, s_{m+1} = q$ of elements of S such that either $s_i \prec s_{i+1}$ or $s_i \succ s_{i+1}$ holds for i, $i = 1, \ldots, m$. A set S is 2-closed, if the conditions (1)–(4) below hold:

S is connected; (1)

when $a, b \in S$ and $a \leqslant c \leqslant b$, then $c \in S$; (2)

when $a, b \in S$, $a \neq b$, and $a \vee b \succ a, b$ then $a \vee b \in S$; (3)

when $a, b \in S$, $a \neq b$, and $a \wedge b \prec a, b$, then $a \wedge b \in S$. (4)

Clearly every convex sublattice of a finite lattice is a 2-closed set, but the converse need not hold. A set S is $2n$-closed if (1), (2), (5) and (6) hold for S:

when $a, b \in S$, $a \neq b$, and there is a maximal chain of length n from a to $a \vee b$ as well as from b to $a \vee b$, then $a \vee b \in S$, where $n = 1, 2, \ldots$; (5)

when $a, b \in S$, $a \neq b$, and there is a maximal chain of length n from a to $a \wedge b$ as well as from b to $a \wedge b$, then $a \wedge b \in S$, where $n = 1, 2, \ldots$. (6)

As above, every convex sublattice of a finite lattice L is $2n$-closed, but the converse need not hold. Further, every 2-closed set is also a $2n$-closed set, where n has the fixed value 1.

2. Characterizations

We characterize first the semimodularity.

THEOREM 1.
A finite lattice L is semimodular if and only if every 2-closed set in L is a convex $\vee$-sublattice of L.

Proof. Assume that every 2-closed set in L is a convex $\vee$-sublattice in L. Let $a, b \in L$ be elements such that $a, b \succ a \wedge b$, and assume now that $a \vee b \not\succ a, b$. Then $S = \{a \wedge b, a, b\}$ is a subset of L where (1), (2) and (4) hold obviously, and because of the assumption above, (3) holds also and $a \vee b \notin S$. But now the 2-closed set S is not a $\vee$-sublattice, which contradicts the basic assumption, and hence the supposition $a \vee b \not\succ a, b$ is false. Thus $a, b \succ a \wedge b$ implies always $a \vee b \succ a, b$, and the semimodularity of L follows.

Let conversely L be semimodular and S a 2-closed set in L. According to (2), S is a convex set. It remains to show that if $a, b \in S$, then also $a \vee b \in S$. Because S is connected, there is a sequence $a = s_1, s_2, \ldots, s_m, s_{m+1} = b$ where either $s_i \prec s_{i+1}$ or $s_i \succ s_{i+1}$, $i = 1, \ldots, m$. If $a \vee b$ is in this sequence, there is nothing to prove. So we assume that $a \vee b$ is not in the sequence. If the sequence contains an element $s > a \vee b$, then $a \vee b \in S$ by (2). Hence we assume further that $s_i < a \vee b$ for every s_i in the sequence. For the simplicity we assume that $a \prec s_2$ and $s_2 \succ s_3 \succ \ldots \succ s_{k-1} \prec s_k$. Because L is semimodular, $s_{k-2} \vee s_k \succ s_k, s_{k-1}$, and because S is 2-closed, $s_{k-2} \vee s_k \in S$. Then, by the semimodularity of L, $s_{k-3} \vee s_{k-2} \vee s_k = s_{k-3} \vee s_k \succ s_k \succ s_{k-3}, s_{k-2} \vee s_k$, and because S is 2-closed, now $s_{k-3} \vee s_k \in S$. After a finite number of similar steps we see that $s_2 \vee s_k \in S$, and there is also a sequence $a = s_1, s_2, s_2 \vee s_k, s_3 \vee s_k, \ldots, s_{k-2} \vee s_k, s_k, s_{k+1}, \ldots, s_m, s_{m+1} = b$ in S, where every two consecutive elements are in the covering relation. By continuing the process, we obtain a sequence with an element $s > a, b$. Then by (2) also $a \vee b \in S$, and thus S is a $\vee$-sublattice. This completes the proof.

By the dual proof we can show that a finite lattice L is dually semimodular if and only if every 2-closed set in L is a convex $\wedge$-sublattice of L. By combining this statement with Theorem 1 we obtain the following corollary, which is valid because a finite lattice is modular if and only if it is semimodular and dually semimodular.

COROLLARY
A finite lattice L is modular if and only if every 2-closed set in L is a convex sublattice in L.

Next we characterize the Jordan-Hölder chain condition.

THEOREM 2.
A finite lattice L satisfies the Jordan-Hölder chain condition if and only if every $2n$-closed set in L is a convex sublattice in L.

Proof. Assume first that every $2n$-closed set in L is a convex sublattice of L. Let a and b be two elements in L, $a < b$, such that there are two maximal chains of unequal lengths between a and b. We can assume further that every two maximal chains between any two elements c and d, $c < d$, in the interval $[a, b]$ where either $a \neq c$ or $d \neq b$, are of equal lengths (or otherwise we substitute $[a, b]$ by $[c, d]$). Let the chains of unequal

lengths between a and b be C_1 and C_2, and let c_1 in C_1 cover a as does also c_2 in C_2. If $c_1 \vee c_2 = b$, then $S = \{a, c_1, c_2\}$ is a $2n$-closed set which is not a convex sublattice of L; a contradiction. If $c_1 \vee c_2 < b$, then the chain $C_1 \setminus \{a\}$ and the maximal chain from c_1 over $c_1 \vee c_2$ to b are of equal lengths. Similarly, $C_2 \setminus \{a\}$ and the maximal chain from c_2 over $c_1 \vee c_2$ to b are of equal lengths as well as the maximal chains from a over c_1 to $c_1 \vee c_2$ and from a over c_2 to $c_1 \vee c_2$. By using these facts, we see that the maximal chains from c_1 over $c_1 \vee c_2$ to b and from c_2 over $c_1 \vee c_2$ to b are of same length, and then also $C_1 \setminus \{a\}$ and $C_2 \setminus \{a\}$ are of the same length. But then C_1 and C_2 are of the same length, which is a contradiction. Thus the supposition that there is in L a pair a, b of elements, $a < b$, between which there are maximal chains of unequal lengths, is false. Hence L satisfies the Jordan-Hölder chain condition.

Let conversely L satisfy the Jordan-Hölder chain condition, and let S be $2n$-closed set in L. According to (2), S is a convex set. Let $a, b \in S$ and $a \neq b$; we shall show that $a \wedge b, a \vee b \in S$. If $a < b$ (or $a > b$), there is nothing to prove. Hence we assume that a and b are noncomparable. Because S is $2n$-closed, it is connected, and so there is a sequence $a = s_1, s_2, \ldots, s_m, s_{m+1} = b$ of elements of S such that either $s_i \prec s_{i+1}$ or $s_i \succ s_{i+1}$, $i = 1, \ldots, m$. Assume for simplicity that $a = s_1 \prec s_2 \succ s_3$. Because L satisfies the Jordan-Hölder chain condition, the maximal chains from s_2 over s_1 to $s_1 \wedge s_3$ and from s_2 over s_3 to $s_1 \wedge s_3$ are of same length. Because $s_1, s_3 \in S$ and there are maximal chains of same length from s_1 to $s_1 \wedge s_3$ and from s_3 to $s_1 \wedge s_3$, and because S is $2n$-closed, $s_1 \wedge s_3 \in S$. By using this technique repeatedly we can show that $a \wedge b \in S$ (and analogously, $a \vee b \in S$), and thus S is a sublattice of L. This completes the proof.

The combination of Corollary and Theorem 2 shows that the Jordan-Hölder chain condition is a generalized modularity in finite lattices.

References

[1] Nieminen J 1985 Distance characterizations in finite lattices *Proc. Indian Acad. Sci. (Math. Sci.)* **94** 43–45.

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Slow magnetohydrodynamic flow past a circular cylinder

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Abstract. Oseen' approximations are used to study the slow motion of a viscous, incompressible, electrically conducting fluid past a circular cylinder in the presence of a uniform aligned magnetic field. Using series truncation method, the analytical solutions for the first three terms in the Fourier sine series expansion of the stream function are obtained. Numerical values of the tangential drag for different values of magnetic interaction parameter and viscous Reynolds number are calculated.

Keywords. Magnetohydrodynamics; circular cylinder; magnetic field; series truncation method.

1. Introduction

The flow of a viscous fluid past spheres and circular cylinders has gained importance in the study of viscous incompressible flow at low and moderate Reynolds numbers. The earlier studies of an electrically non-conducting fluid are due to several workers (e.g. see [6, 7, 10–11, 13–16]). Some of these solutions may also be found in the book edited by Rosenhead [12].

Due to the increasing technical applications of magnetohydrodynamic (MHD) effects, the flow of an electrically conducting viscous fluid past axisymmetric bodies in the presence of magnetic field at low values of viscous Reynolds number and magnetic Reynolds number has drawn much attention in recent years. MHD-Stokes' and Oseen' flow past a sphere has been studied by many workers (see e.g. [1, 2, 5, 8, 9]. Some of these solutions have been summarized by Cabannes [4].

Yosinobu and Kakutani [17] applied Chester's method for MHD flow past a circular cylinder in aligned and transverse magnetic fields and obtained the drag and lift coefficients for small Hartmann number.

In a recent paper Bramley [3] applied the Oseen' approximations to study MHD flow past a circular cylinder in an aligned magnetic field using the series truncation method suggested by Underwood [15] and obtained the solution for the first two terms in the series expansion. However, it was noticed that his MHD results do not reduce to the non-magnetic case when the magnetic interaction parameter is taken as zero. Moreover, there seems to be a misprint in the expression of tangential drag.

In the present paper we have reconsidered this problem and obtained analytical solutions upto third order terms and the correct expression for the coefficient of tangential drag is obtained.

2. Formulation of the problem

We consider the streaming motion of an incompressible, viscous, electrically conducting fluid past a circular cylinder. At infinity, the streaming motion and the magnetic field are assumed to be uniform and their directions are parallel to the positive x'-axis (figure 1).

The flow is assumed to be steady and polar coordinates are used with the angle θ measured from the upstream axis. The equations governing the motion are

$$\text{Curl } \mathbf{H}' = \mathbf{J}', \text{ div } \mathbf{H}' = 0, \tag{1}$$

$$\text{Curl } \mathbf{E}' = 0, \tag{2}$$

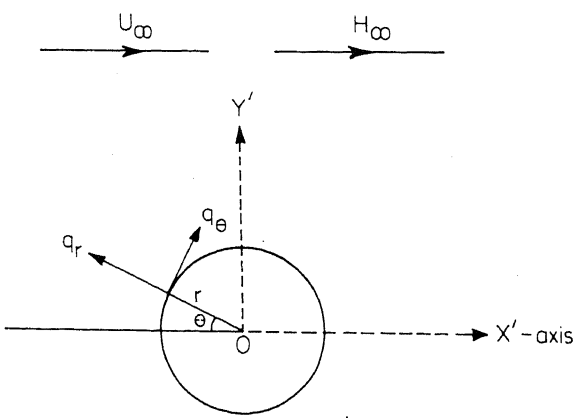
$$\mathbf{J}' = \sigma(\mathbf{E}' + \mu_e \mathbf{v}' \times \mathbf{H}'), \tag{3}$$

$$\text{div } \mathbf{v}' = 0, \tag{4}$$

$$\rho \text{ grad } (\tfrac{1}{2}\mathbf{v}'\cdot\mathbf{v}') - \rho\mathbf{v}' \times \text{curl } \mathbf{v}', = -\text{grad } p' + \mu\nabla^2\mathbf{v}' + \mu_e \mathbf{J}' \times \mathbf{H}', \tag{5}$$

Table 1. Tangential drag coefficient in MHD-Oseen flow for various values of magnetic interaction parameter m and for two values of viscous Reynolds number

	$R = 0.05$	$R = 0.1$
m	C_{D_T}	C_{D_T}
0.0	2.008456024	2.550985477
0.0025	2.010817481	2.551901876
0.005	2.017964565	2.554651770
0.025	2.205266445	2.639986677
0.05	2.528405471	2.852194843
0.125	3.371606384	3.601393559
0.25	4.514485145	4.734618623
0.375	5.550688280	5.786941974
0.5	6.601510363	6.870869312



where, $\mathbf{v}'$ is the fluid velocity, $\mathbf{H}'$ is the magnetic field, $\mathbf{E}'$ is the electrical field, $\mathbf{J}'$ is the current vector, σ is the electrical conductivity, ρ is the mass density, μ_e is the magnetic permeability, and μ is the viscosity of the fluid. The differentiations are taken with respect to the physical variables $\mathbf{r}'$, θ. From the first three equations we deduce that

$$\mu_e \sigma \text{ curl } (\mathbf{v}' \times \mathbf{H}') = \text{ curl curl } \mathbf{H}'. \tag{6}$$

Let U_∞ be the speed of the uniform stream at infinity parallel to x'-axis, and let a be the radius of the cylinder. The boundary conditions are

$$\left.\begin{aligned} &r' = a : \mathbf{v}' = 0 \\ &r' = \infty : \mathbf{v}' = U_\infty \hat{i},\ p' = 0, \end{aligned}\right\} \tag{7}$$

where $\hat{i}$ is the unit vector along $0x'$. Let us introduce the following non-dimensional quantities

$$\begin{aligned} &\nabla = a\nabla',\ p = p'a/\mu U_\infty,\ r = r'/a,\ \mathbf{v} = \mathbf{v}'/U_\infty, \\ &\mathbf{H} = \mathbf{H}'/H_\infty, \mathbf{J} = \mathbf{J}'/(U_\infty \mu_e \sigma H_\infty), \end{aligned} \tag{8}$$

where H_∞ is the intensity of the applied aligned external magnetic field. Equations (4), (5) and (6) in non-dimensional form become

$$\nabla \cdot \mathbf{v} = 0, \tag{9}$$

$$\frac{\mathrm{Re}}{2}[\nabla(\tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}) - \mathbf{v} \times (\nabla \times \mathbf{v})] = -\nabla p + \nabla^2 \mathbf{v} + \mathrm{Re}_h^2 \mathbf{J} \times \mathbf{H} \tag{10}$$

$$\mathrm{Re}_\sigma \nabla \times (\mathbf{v} \times \mathbf{H}) = \nabla \times (\nabla \times \mathbf{H}) \tag{11}$$

where $\mathrm{Re} = \dfrac{2\rho U_\infty a}{\mu}$ (viscous Reynolds number),

$\mathrm{Re}_h = \mu_e H_\infty a(\sigma/\mu)^{\frac{1}{2}}$ (Hartmann number),

$\mathrm{Re}_\sigma = \mu_e \sigma a U_\infty$ (magnetic Reynolds number). (12)

The magnetic field and the fluid flow are aligned at infinity so the electric field can be assumed to be zero throughout the flow field

From (3), we get

$$\mathbf{J} = \mathbf{v} \times \mathbf{H} \tag{13}$$

If Re_σ is small, then from (11), we have

$$\nabla \times (\nabla \times \mathbf{H}) = 0. \tag{14}$$

This shows that the magnetic field is independent of the fluid velocity. Hence

$$\mathbf{H} = \hat{i} = (-\cos\theta, \sin\theta, 0). \tag{15}$$

Substituting (13) in (10) and taking curl of both sides, we get

$$\frac{\mathrm{Re}}{2} \nabla \times (\omega \times \mathbf{v}) = \nabla^2 \omega + \mathrm{Re}_h^2 \nabla \times \{(\mathbf{v} \times \mathbf{H}) \times \mathbf{H}\}, \tag{16}$$

where $\omega = \nabla \times \mathbf{v}$. (17)

Introducing the dimensionless stream function $\psi(r, \theta)$, such that

$$\mathbf{v} = \left(-\frac{1}{r}\frac{\partial \psi}{\partial \theta}, \frac{\partial \psi}{\partial r}, 0 \right), \tag{18}$$

the equation of continuity (9) is identically satisfied and the equation of motion (16), becomes

$$\frac{\mathrm{Re}}{2}\left[-\frac{1}{r}\frac{\partial \psi}{\partial \theta}\frac{\partial}{\partial r}(\nabla^2\psi) + \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial}{\partial \theta}(\nabla^2\psi) \right] = \nabla^4\psi$$
$$-\mathrm{Re}_h^2\left[\sin^2\theta\,\nabla^2\psi - \sin 2\theta\left(\frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial \psi}{\partial \theta} \right) + \cos 2\theta \frac{\partial^2 \psi}{\partial r^2} \right]. \tag{19}$$

Applying Oseen' approximations on the left side terms, i.e. in the inertia terms of equation (19), keeping in view that at infinity, $\psi \to r \sin\theta$, we get the Oseen' equation of motion as

$$\frac{\mathrm{Re}}{2}\left[-\cos\theta\frac{\partial}{\partial r}(\nabla^2\psi) + \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}(\nabla^2\psi) \right] = \nabla^4\psi$$
$$-\mathrm{Re}_h^2\left[\sin^2\theta\,\nabla^2\psi - \sin 2\theta\left(\frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\psi}{\partial\theta} \right) + \cos 2\theta\frac{\partial^2\psi}{\partial r^2} \right], \tag{20}$$

and the boundary conditions are

$$\left.\begin{aligned} &r = 1 : \psi = 0,\ \partial\psi/\partial r = 0 \\ &r = \infty : \partial\psi/\partial r \to \sin\theta \end{aligned}\right\} \tag{21}$$

3. Method of solution

For the solution of (20), the stream function is expanded in a Fourier sine series

$$\psi(r, \theta) = \sum_{n=1}^{\infty} f_n(r) \sin n\theta. \tag{22}$$

Substituting (22) in (20) and equating the coefficients of $\sin\theta, \sin 2\theta, \sin 3\theta$ and applying series truncation method, i.e. $f_4(r)$ and later terms in the Fourier series for the stream function are assumed to be zero, we get the following ordinary differential equations for $f_1(r)$, $f_2(r)$ and $f_3(r)$:

$$f_1'''' + \frac{2}{r}f_1''' - \frac{3}{r^2}f_1'' + \frac{3}{r^3}f_1' - \frac{3}{r^4}f_1$$
$$+ R\left[f_2''' + \frac{f_2''}{r} - \frac{5}{r^2}f_2' + \frac{8}{r^3}f_2 + \frac{2}{r}\left(f_2'' + \frac{f_2'}{r} - \frac{4}{r^2}f_2 \right) \right]$$
$$- m^2\left[\left(f_1'' + \frac{f_1'}{r} - \frac{f_1}{r^2} \right) + \left(f_3'' + \frac{5}{r}f_3' + \frac{3}{r^2}f_3 \right) \right] = 0, \tag{23}$$
$$f_2'''' + \frac{2}{r}f_2''' - \frac{9}{r^2}f_2'' + \frac{9}{r^3}f_2'$$

$$+R\left[f_1'''+\frac{f_1''}{r}-\frac{2}{r^2}f_1'+\frac{2}{r^3}f_1-\frac{1}{r}\left(f_1''+\frac{f_1'}{r}-\frac{f_1}{r^2}\right)\right.$$
$$\left.+f_3'''+\frac{f_3''}{r}-\frac{10}{r^2}f_3'+\frac{18}{r^3}f_3+\frac{3}{r}\left(f_3''+\frac{f_3}{r}-\frac{9}{r^2}f_3\right)\right]$$
$$-2m^2\left(f_2''+\frac{f_2'}{r}-\frac{4}{r^2}f_2\right)=0, \tag{24}$$

$$f_3''''+\frac{2}{r}f_3'''-\frac{19}{r^2}f_3''+\frac{19}{r^3}f_3'+\frac{45}{r^4}f_3$$
$$+R\left[f_2'''+\frac{f_2''}{r}-\frac{5}{r^2}f_2'+\frac{8}{r^3}f_2-\frac{2}{r}\left(f_2''+\frac{f_2'}{r}-\frac{4}{r^2}f_2\right)\right]$$
$$-m^2\left[f_1''-\frac{3}{r}f_1'+\frac{3}{r^2}f_1+2\left(f_3''+\frac{f_3'}{r}-\frac{9}{r^2}f_3\right)\right]=0, \tag{25}$$

where $m^2=\mathrm{Re}_h^2/4$ and $R=\mathrm{Re}/4$. (26)

The boundary conditions are

$$r=1: f_1=f_1'=0, f_2=f_2'=0, f_3=f_3'=0,$$
$$r=\infty: f_1'=1, f_2'=0, f_3'=0. \tag{27}$$

4. OFD-Oseen flow

In the absence of the magnetic field, putting $\mathrm{Re}_h=0$, equations (23) to (25) reduce to

$$f_1''''+\frac{2}{r}f_1'''-\frac{3}{r^2}f_1''+\frac{3}{r^3}f_1'-\frac{3}{r^4}f_1$$
$$+R\left[f_2'''+\frac{f_2''}{r}-\frac{5}{r^2}f_2'+\frac{8}{r^3}f_2+\frac{2}{r}\left(f_2''+\frac{f_2'}{r}-\frac{4}{r^2}f_2\right)\right]=0, \tag{28}$$

$$f_2''''+\frac{2}{r}f_2'''-\frac{9}{r^2}f_2''+\frac{9}{r^3}f_2'$$
$$+R\left[f_1'''+\frac{f_1''}{r}-\frac{2}{r^2}f_1'+\frac{2}{r^3}f_1-\frac{1}{r}\left(f_1''+\frac{f_1'}{r}-\frac{f_1}{r^2}\right)\right.$$
$$\left.+f_3'''+\frac{f_3''}{r}-\frac{10}{r^2}f_3'+\frac{18}{r^3}f_3+\frac{3}{r}\left(f_3''+\frac{f_3'}{r}-\frac{9}{r^2}f_3\right)\right]=0, \tag{29}$$

$$f_3''''+\frac{2}{r}f_3'''-\frac{19}{r^2}f_3''+\frac{19}{r^3}f_3'+\frac{45}{r^4}f_3$$
$$+R\left[f_2'''+\frac{f_2''}{r}-\frac{5}{r^2}f_2'+\frac{8}{r^3}f_2-\frac{2}{r}\left(f_2''+\frac{f_2'}{r}-\frac{4}{r^2}f_2\right)\right]=0 \tag{30}$$

Let us take

$$f_1'' + \frac{f_1'}{r} - \frac{f_1}{r^2} = g_1, \tag{31}$$

$$f_2'' + \frac{f_2'}{r} - \frac{4}{r^2} f_2 = g_2, \tag{32}$$

$$f_3'' + \frac{f_3'}{r} - \frac{9}{r^2} f_3 = g_3, \tag{33}$$

then (28), (29) and (30) become

$$g_1'' + \frac{1}{r} g_1' - \frac{1}{r^2} g_1 + R\left(g_2' + \frac{2}{r} g_2\right) = 0, \tag{34}$$

$$g_2'' + \frac{g_2'}{r} - \frac{4}{r^2} g_2 + R\left[\left(g_1' - \frac{g_1}{r}\right) + \left(g_3' + \frac{3}{r} g_3\right)\right] = 0 \tag{35}$$

$$g_3'' + \frac{g_3'}{r} - \frac{9}{r^2} g_3 + R\left[g_2' - \frac{2}{r} g_2\right] = 0. \tag{36}$$

Further, taking

$$g_1' - \frac{1}{r} g_1 = h_1, \tag{37}$$

$$g_2' + \frac{2}{r} g_2 = h_2, \tag{38}$$

$$g_3' + \frac{3}{r} g_3 = h_3, \tag{39}$$

(34), (35) and (36) reduce to

$$h_1' + \frac{2}{r} h_1 + R h_2 = 0, \tag{40}$$

$$h_2' - \frac{1}{r} h_2 + \mathrm{R h_1} = -\mathrm{R h_3}, \tag{41}$$

$$h_3' + \frac{2}{r} h_3 + R h_2 = \frac{4}{r} (R g_2 + h_3). \tag{42}$$

These are coupled differential equations and their solutions may be obtained by the method of successive approximations. Their first approximations, satisfying the boundary condition at infinity may be written as

$$h_1 = A R^3 K_2 (rR), \tag{43}$$

$$h_2 = B R^3 K_1 (rR), \tag{44}$$

$$h_3 = B R^3 K_0 (rR) + \frac{2BR^2}{r} K_1 (rR) = B R^3 K_2 (rR), \tag{45}$$

where A and B are arbitrary constants. Substituting (43) and (44) in (40) and using the

recurrence relation viz

$$x K_n'(x) = -n K_n(x) - x K_{n-1}(x), \tag{46}$$

we find that

$$A = B. \tag{47}$$

It may be noted that if $h_3 = 0$ then (43) and (44) are exact solutions of equations (40) and (41) respectively. The corresponding solutions for g_1 and g_2 from (37) and (38) satisfying the boundary condition at infinity, are obtained as:

$$g_1 = -AR^2 K_1(rR), \tag{48}$$

$$g_2 = -AR^2 K_2(rR) - \frac{4C_2}{r^2}, \tag{49}$$

$$g_3 = -AR^2 K_1(rR) - \frac{4ARK_2(rR)}{r} + \frac{D}{r^3}, \tag{50}$$

where C_2 and D are arbitrary constants.

Finally from (31) to (33), the solutions for f_1, f_2 and f_3 are obtained as:

$$f_1 = r + \frac{C_1}{r} - AK_1(rR), \tag{51}$$

$$f_2 = C_2 + \frac{C_3}{r^2} - AK_2(rR), \tag{52}$$

$$f_3 = -\frac{D}{8r} + \frac{C_4}{r^3} - AK_3(rR) \tag{53}$$

Applying the boundary conditions (27), the unknown constants are obtained as

$$A = -2/RK_0(R), \tag{54}$$

$$C_1 = -1 + AK_1(R), \tag{55}$$

$$C_2 = K_1(R)/K_0(R), \tag{56}$$

$$C_3 = -C_2 + AK_2(R), \tag{57}$$

$$C_4 = -\left(1 + \frac{8}{R^2}\right) - \frac{4}{R}\left(1 + \frac{4}{R^2}\right) C_2, \tag{58}$$

$$D = 8C_4 - 8AK_3(R). \tag{59}$$

5. MHD-Oseen' flow

Guided by the solution of §4, we now try the solutions of (23) to (25) in the following form:

$$f_1(r) = r + \frac{C_1}{r} - F_1(r), \tag{60}$$

$$f_2(r) = C_2 + \frac{C_3}{r^2} - F_2(r), \tag{61}$$

$$f_3(r) = \frac{-D}{8r} + \frac{C_4}{r^3} - F_3(r), \tag{62}$$

and look for a solution where

$$F_1(r) = AK_1(rP),\ F_2(r) = AQK_2(rP),\ F_3(r) = ASK_3(rP), \tag{63}$$

A, Q, S, P being constants. Substituting (63) in (23) to (25) and comparing the coefficients of $K_1(rP)$ and $K_0(rP)$, we get

$$P^2 - RQP - m^2(1+S) = 0, \tag{64}$$

$$QP^2 - 2m^2Q - RP(1+S) = 0, \tag{65}$$

$$SP^2 - RQP - m^2(1+2S) = 0. \tag{66}$$

Proceeding parallel to the analysis of §4, the first order approximation is obtained if we substitute $S = 0$ in (64) and (65) and after eliminating Q, we get

$$P^2 = \tfrac{1}{2}\{(3m^2+R^2) \pm [(3m^2+R^2)^2 - 8m^4]^{1/2}\} \tag{67}$$

Since $F_1(r), F_2(r), F_3(r)$ must be small for large r, the two solutions of (67), which satisfy this condition, are

$$\begin{aligned} P_1 &= [\tfrac{1}{2}\{(3m^2+R^2) + [(3m^2+R^2)^2 - 8m^4]^{1/2}\}]^{1/2} \\ P_2 &= [\tfrac{1}{2}\{(3m^2+R^2) - [(3m^2+R^2)^2 - 8m^4]^{1/2}\}]^{1/2} \end{aligned} \tag{68}$$

and

$$Q_1 = (P_1^2 - m^2)/RP_1, \qquad Q_2 = (P_2^2 - m^2)/RP_2, \tag{69}$$

$$S_1 = P_1^2/(P_1^2 - 2m^2), \qquad S_2 = P_2^2/(P_2^2 - 2m^2). \tag{70}$$

The solutions of equations (23) to (25), satisfying the boundary conditions at infinity, are

$$f_1(r) = r + \frac{C_1}{r} - A_1K_1(rP_1) - A_2K_1(rP_2), \tag{71}$$

$$f_2(r) = C_2 + \frac{C_3}{r^2} - A_1Q_1K_2(rP_1) - A_2Q_2K_2(rP_2), \tag{72}$$

$$f_3(r) = -\frac{D}{8r} + \frac{C_4}{r^3} - A_1S_1K_3(rP_1) - A_2S_2K_3(rP_2), \tag{73}$$

where C_2 is the same as given in (56). The other constants will now be modified by the presence of the *magnetic interaction parameter* m and may be obtained, by applying the boundary conditions at $r = 1$, as

$$A_1 = \frac{2[Q_2K_1(P_2) - C_2K_0(P_2)]}{P_1[Q_1K_1(P_1)K_0(P_2) - Q_2K_0(P_1)K_1(P_2)]}, \tag{74}$$

$$A_2 = -\frac{2C_2 + A_1Q_1P_1K_1(P_1)}{Q_2P_2K_1(P_2)}, \tag{75}$$

$$C_1 = -1 + A_1 K_1(P_1) + A_2 K_1(P_2), \tag{76}$$

$$C_3 = A_1 Q_1 K_2(P_1) + A_2 Q_2 K_2(P_2), \tag{77}$$

$$D = 4A_1 S_1 P_1 K_2(P_1) + 4A_2 S_2 P_2 K_2(P_2), \tag{78}$$

$$C_4 = \frac{D}{8} + A_1 S_1 K_3(P_1) + A_2 S_2 K_3(P_2). \tag{79}$$

It may be noted that when m tends to zero, we get

$$P_1 = R,\ P_2 = 0,\ Q_1 = 1,\ Q_2 = 0,\ S_1 = 1,\ S_2 = 1,$$

$$A_1 = -\frac{2}{RK_0(R)},\ A_2 = 0,$$

then the above results reduce to that of §4.

6. Drag

The tangential component of the viscous stress tensor on the surface of the cylinder is given by

$$[\tau_{r\theta}]_{r=1} = \frac{\mu U_\infty}{a}\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}\right]_{r=1}$$

$$= \frac{\mu U_\infty}{a}[4f_3''(1)\cos^2\theta + 2f_2''(1)\cos\theta$$

$$+ \{f_1''(1) - f_3''(1)\} + \ldots]\sin\theta \tag{80}$$

Therefore, the tangential drag on the cylinder of unit length is obtained by

$$D_T = \int_0^{2\pi} (\tau_{r\theta})_{r=1} \sin\theta\, a\, \mathrm{d}\theta$$

$$= \mu U_\infty \pi f_1''(1). \tag{81}$$

It may be noted that in this only the first term of the series expansion contributes. The coefficient of tangential drag is obtained as

$$C_{D_T} = \frac{D_T}{\mu U_\infty} = \pi f_1''(1), \tag{82}$$

where $f_1''(1) = -[2 + A_1 P_1 K_0(P_1) + A_2 P_2 K_0(P_2)$

$$+ A_1 P_1^2 K_1(P_1) + A_2 P_2^2 K_1(P_2)] \tag{83}$$

This shows that there is a misprint in the paper of Bramley [3] in reporting the above expression.

7. Numerical discussion

The coefficient of tangential drag (C_{D_T}) is calculated for various values of the magnetic interaction parameter m and for two values of viscous Reynolds number Re, viz 0.05

and 0·1. The results are tabulated in table 1. It is noted that the value of C_{D_T} increases with the magnetic interaction parameter m.

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References

[1] Bansal J L and Rama Kumari MHD slow motion past a sphere, *Proc. Indian Acad. Sci.* (Math. Sci.) **92** (1983), 157–166

[2] Blerkom R Von *Magnetohydrodynamic flow of a viscous fluid past a sphere*, Ph.D. Thesis, Harvard University (1959) *J. Fluid Mech.* **8** (1960), 432–441

[3] Bramley J S Magnetohydrodynamic flow past a circular cylinder, *Z. Angew. Math. Phys.* **25** (1974), 409–415

[4] Cabannes H *Theoretical magnetofluiddynamics*, Academic Press, New York, 1970.

[5] Chester W The effect of a magnetic field on Stokes flow in a conducting fluid, *J. Fluid Mech.* **3** (1957), 304–308

[6] Kaplun S Low Reynolds number flow past a circular cylinder, *J. Math. Mech.* **6** (1957), 595–603

[7] Lamb H On the uniform motion of a sphere in a viscous fluid, *Philos. Mag.* **21** (1911), 112–119

[8] Ludford G S S The effect of an aligned magnetic field on Oseen flow of a conducting fluid, *Arch. Rational Mech. Anal.* **4** (1959), 405–411.

[9] Mathon R and Ranger K B Magnetohydrodynamic streaming flow past a sphere at low Hartmann numbers, *Phys. Fluids*, **16** (1973), 485–490

[10] Oseen C W Über die Stokessche Formel und Über eine Verwandte Anfgabe in der Hydrodynamik, *Arkiv. Mat. Astron. Fys.* **6** (1910)

[11] Proudman I and Pearson J R A Expansions at small Reynolds numbers for the flow past a sphere and a circular cylinder, *J. Fluid Mech.* **2** (1957), 237–262

[12] Rosenhead L (ed) *Laminar boundary Layers.* Oxford University Press (1963)

[13] Stokes G G On the effect of the internal friction of fluids on the motion of pendulums, *Trans. Cambridge Philos. Soc.* **9** (1851), 8–106

[14] Tomotika S and Aoi T The steady flow of a viscous fluid past a sphere and circular cylinder at small Reynolds numbers, *Q. J. Mech. Appl. Math.* **3** (1950), 140–161

[15] Underwood R L Calculation of incompressible flow past a circular cylinder at moderate Reynolds numbers, *J. Fluid Mech.* **37** (1969), 95–114

[16] Yamada H On the slow motion of a viscous fluid past a circular cylinder, *Rep. Res. Inst. Appl. Mech. Kyushu Univ.*, **3** (1954), 11–23

[17] Yosinobu H and Kakutani T *J. Phys. Soc. Jpn.* **14** (1959), 1433–1444

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On Extended Hermite-Fezér interpolation based on the zeros of Laguerre polynomials

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Abstract. Szász [2] proved the existence of an interpolatory polynomial which interpolates a given function at given data with some extended points. He also gave its explicit representation. We study the uniform convergence of the same, interpolating at the zeros of Laguerre polynomials with one point (as origin) extension.

Keywords. Interpolation; polynomial approximation; uniform norm; Laguerre polynomials.

1. Introduction

Let there be two disjoint sets of points $\{x_{kn}\}$ $(k = 1, 2, \ldots, n)$ and $\{\xi_{im}\}$, $(i = 1, 2 \ldots, m)$ in the real interval $[A, B]$. $C[A, B]$ is a linear space of continuous functions defined over $[A, B]$. Then it is known [2] that there exists a uniquely determined polynomial $S_N(x)$ of degree $N \leqslant 2n+m-1$. $S_N(x) \in P(x)$ where $P(x)$ is a linear subspace of $C[A, B]$ consisting of all polynomials. $S_N(x)$ satisfies the following conditions:

$$S_N(x_k) = f(x_k),\ S'_N(x_k) = d_k\ (k = 1, \ldots n,\ x_k \equiv x_{kn},\ d_k \equiv d_{kn}), \tag{1}$$

$$S_N(\xi_i) = f(\xi_i),\ (\xi_{im} \equiv \xi_i,\ i = 1, 2 \ldots m). \tag{2}$$

d_k $(k = 1, \ldots n)$ are real numbers. Also $S_N(x)$ is given by

$$\begin{aligned} S_N(x) &= \sum_{k=1}^{n} f(x_k) \frac{\Omega_m(x)}{\Omega_m(x_k)} [1 + c_k(x - x_k)]\, l_k^2(x) \\ &\quad + \sum_{i=1}^{n} f(\xi_i) \frac{\omega_n^2(x)}{\omega_n^2(\xi_i)} L_i(x) + \sum_{k=1}^{n} d_k \frac{\Omega_m(x)}{\Omega_m(x_k)} (x - x_k) l_k^2(x) \\ &\overset{\text{def}}{=} \sum_{k=1}^{n} f(x_k) h_k(x) + \sum_{i=1}^{m} f(\xi_i) t_i(x) + \sum_{k=1}^{n} d_k H_k(x), \end{aligned} \tag{3}$$

where

$$\omega_n(x) = C_n(x - x_1) \ldots (x - x_n),\ C_n \neq 0; \tag{4}$$

$$\Omega_m(x) = K_m(x - \xi_1) \ldots (x - \xi_m),\ K_m \neq 0; \tag{5}$$

$$c_k = \sum_{i=1}^{n} \frac{1}{\xi_i - x_k} - \frac{\omega_n''(x_k)}{\omega_n'(x_k)},\ (k = 1, 2, \ldots n); \tag{6}$$

$$l_k(x) = \frac{\omega_n(x)}{\omega_n'(x_k)\,(x-x_k)}; \tag{7}$$

$$L_i(x) = \Omega_m(x)\,[\Omega'(\xi_i)\,(x-\xi_i)]^{-1}. \tag{8}$$

$S_N(x) \equiv S_N(f, d, x)$, $(N \leqslant 2n+m-1)$ called extended Hermite-Fezér interpolation polynomial, has interpolatory property at the roots of $\omega_n(x)$ and $\Omega_m(x)$. Replacing $\omega_n(x)$ by Laguerre polynomials $L_n^{(\alpha)}(x)$ and denoting by $x_{kn}\,(\equiv x_{kn}^{(\alpha)}, k = 1, \ldots, n)$ the kth root of $L_n^{(\alpha)}(x)$, from (3) we get for $m = 1$, $\xi_1 = 0$; $d_k = 0$, $k = 1, \ldots, n$;

$$S_N(x) \overset{\text{def}}{=} A_n^{(\alpha)}(f, x) = \left[\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}\right]^2 f(0) + \sum_{k=1}^{n} f(x_k)\frac{x}{x_k}\, v_k^{(\alpha)}(x)\left\{\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)\prime}(x_k)\,(x-x_k)}\right\}^2, \tag{9}$$

where

$$v_k^{(\alpha)}(x) = 1 - \left\{\frac{1}{x_k} + \frac{L_n^{(\alpha)\prime\prime}(x_k)}{L_n^{(\alpha)\prime}(x_k)}\right\}(x-x_k). \tag{10}$$

$A_n^{(\alpha)}(f, x)$ is called A-quasi-Hermite-Fezér interpolation polynomial of degree $\leqslant 2n$. From differential equation

$$xy'' + (\alpha+1-x)\,y' + ny = 0,\ y \equiv L_n^{(\alpha)}(x) \text{ (see [3])}; \tag{11}$$

We have

$$v_k^{(\alpha)}(x) = 1 - \frac{(x_k-\alpha)\,(x-x_k)}{x_k}. \tag{12}$$

2. Results

Let $C(m) = \{f(x)$: f is continuous in $[0, \infty)$, $f(x) = O(x^m)$ as $x \to \infty$; $m \geqslant 0$ is an integer.$\}$

Consider the Hermite-Fezér interpolation polynomial

$$H_n^{(\alpha)}(f, x) = \sum_{k=1}^{n} f(x_k)\, v(x)\, l_k^2(x),\ \alpha > -1; \tag{13}$$

where

$$v\ (x) = \frac{(x-x_k)\,(\alpha-x_k)+x}{x_k},\ (k = 1, 2 \ldots, n).$$

Then the first convergence theorem by Szegö ([3] theorem 14.7) is

$$\lim_{n\to\infty} \|f(x) - H_n^{(\alpha)}(f, x)\|_I = 0 \tag{14}$$

$\forall f \in C(s)$ and $I \subset (0, \infty)$ for $\alpha \geqslant 0$, or, $I \subset [0, \infty)$ for $-1 < \alpha < 0$. Furthermore there exists a function in $C(m)$ such that $\{H_n^{(\alpha)}(f, x)\}$ diverges for $\alpha \geqslant 0$ at $x = 0$. As for the rate of the convergence the following result is due to Vertesi [4]

$$\|f(x) - H_n^{(\alpha)}(f, x)\|_I = \begin{cases} O(\omega(f, n^{-1-\alpha})); & -1 < \alpha < -\frac{1}{2} \\ O\left(\omega\left(f, \dfrac{\log n}{\sqrt{n}}\right)\right); & \alpha \geqslant -\frac{1}{2}, \end{cases} \tag{15}$$

$\forall f \in C(m)$ where $I \subset (0, \infty)$ and $\omega(f, h)$ is modulus of continuity in $[0, \Delta) \supset I$. Again $\forall f \in C(s)$ the following result is due to Szabados [1]

$$\|f(x) - H_n^{(\alpha)}(f, x)\|_{[0,\Delta]} = \begin{cases} O\left(\omega\left(f, \frac{\log n}{\sqrt{n}}\right)\right); -1 < \alpha \leqslant -\frac{1}{2} \\ O(\omega(f, n^{\alpha} \log n)); -\frac{1}{2} \leqslant \alpha < 0. \end{cases} \tag{16}$$

Our aim is to construct such interpolatory polynomial which converges to $f(x)$ in $[0, \Delta]$, $\Delta < \infty$ for $\alpha > -1$. However we conclude the following important result

$$\|f(x) - A_n^{(\alpha)}(f, x)\|_{[0,\Delta]} = \begin{cases} O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_{\Delta}(i/\sqrt{n}) i^{2\alpha-3}\right), -\frac{1}{2} \leqslant \alpha < 1; \\ O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_{\Delta}(i/\sqrt{n}) i^{-2}\right), -1 < \alpha < -\frac{1}{2}; \end{cases} \tag{17}$$

where $\omega_{\Delta}(h)$ is the modulus of continuity on $[0, \Delta + \varepsilon]$ such that $\omega_{\Delta}(f, h) = O[\omega_{\Delta}(h)]$ Thus we prove the following:

THEOREM 1.

$$f \in C(m) \Rightarrow (17), \textit{ holds.} \tag{18}$$

For the lower estimation, we have

THEOREM 2. *Let* $\lim_{t \to 0} \omega(t) t^{-1} = \infty$ *then* $\exists f \in C(m)$:

$$\|f(x) - A_n^{(\alpha)}(f, x)\|_{[0,\Delta]} > \begin{cases} \sum_{i=1}^{[\sqrt{n}]} \omega_{\Delta}(i/\sqrt{n}) i^{2\alpha-3}, -\frac{1}{2} \leqslant \alpha < 1; \\ \sum_{i=1}^{[\sqrt{n}]} \omega_{\Delta}(i/\sqrt{n}) i^{-2}, -1 < \alpha < -\frac{1}{2}, \end{cases} \tag{19}$$

The following results shall be used in the sequel for $\alpha > -1$.

$$2\sqrt{x_{kn}^{(\alpha)}} = \frac{1}{\sqrt{n}}[k\pi + O(1)], (0 < x_{kn}^{(\alpha)} \leqslant \Omega, n = 1, 2 \ldots). \tag{20}$$

$$|L_n^{(\alpha)\prime}(x_k)| \sim k^{-\alpha-3/2} n^{\alpha+1}, (0 < x_k \equiv x_{kn}^{(\alpha)} \leqslant \Omega, n = 1, 2 \ldots). \tag{21}$$

$$|L_n^{(\alpha)}(x)| = \begin{cases} x^{-\alpha/2-1/4} O(n^{\alpha/2-1/4}), cn^{-1} \leqslant x \leqslant \Omega & (22) \\ O(n^{\alpha}), 0 \leqslant x \leqslant cn^{-1} & (23) \end{cases}$$

$$\sum_{k=1}^{n} x_k^{m-1} [L_n^{(\alpha)\prime}(x_k)]^{-2} = \frac{\Gamma(n+1)\Gamma(m+\alpha+3)}{\Gamma(n+\alpha+1)} \sim n^{-\alpha},$$

$(m = 1, 2 \ldots, 2n-3)$.

$$\frac{d}{dx}\{L_n^{(\alpha)}(x)\} = -L_{n-1}^{\alpha+1}(x). \tag{24}$$

References for (20) to (24) is [3]. We also use the following elementary results [4]. Let x_j be the nearest root to x, then for $k \neq j$, $k = j \pm i$, $0 \leqslant x, x_k \leqslant \Omega$ we have

$$|x - x_k| = O(1)\frac{ij + i^2}{n} \tag{25}$$

$$(x_{k+1} - x_k) \sim \begin{cases} 1/n, & k = O(1) \\ 1/\sqrt{n}, & \varepsilon \leqslant x_k, x_{k+1} \leqslant \Omega. \end{cases} \tag{26}$$

$$|f(x) - f(x_j)| = \begin{cases} O(\omega_\Delta(1/n)), & x = 0 \\ O(1)\,\omega_\Delta(1/\sqrt{n}), & \delta \leqslant x, x_j \leqslant \Delta + \varepsilon \end{cases} \tag{27}$$

$$0 < \xi(\alpha) \leqslant v_k^{(\alpha)}(x) \leqslant M(\alpha) < \infty; \tag{28}$$

for $|x - x_k| \leqslant \xi(\alpha) \leqslant \varepsilon, \alpha < 1; \varepsilon > 0$ is arbitrary and fixed. Also we use without reference that $j \sim 1$ for $0 \leqslant x \leqslant c/n; j \sim \sqrt{n}$ and $k \sim \sqrt{n}$ for $0 < \delta \leqslant x, x_k \leqslant \Omega$ so that we have $i \sim k$ in $0 \leqslant x \leqslant \Delta + \varepsilon$ and $x_k \leqslant \Delta + \varepsilon$.

From (22), we have

$$\sqrt{x}\, L_n^{(\alpha+1)}(x) \sim \sqrt{n}\, L_n^{(\alpha)}(x) \text{ for } x \in J, \tag{29}$$

where

$$J = [c/n, \Delta + \varepsilon] \setminus \left\{ \bigcup_{k=1}^{n} [x_{k,n}^{(\alpha+1)} - c_1/n, x_{kn}^{(\alpha+1)} + c_1/n] \right.$$
$$\left. \bigcup \left\{ \bigcup_{k=1}^{n} [x_{kn}^{(\alpha)} - c_2/n, x_{kn}^{(\alpha)} + c_2/n] \right\} \right\}. \tag{30}$$

Moreover (29) holds in J^* where

$$J^* = [0, c/n] \setminus \{0\}. \tag{31}$$

3. Proofs

3.1 *Proof of theorem* 1.

From $A_n^{(\alpha+1)}(1, x) \equiv 1$ we have

$$|f(x) - A_n^{(\alpha+1)}(f, x)| \leqslant |f(x) - f(0)| \left[\frac{L_n^{(\alpha+1)}(x)}{L_n^{(\alpha+1)}(0)} \right]^2$$
$$+ \sum_{k=1}^{n} |f(x) - f(x_k)|\, |v_k^{(\alpha+1)}(x)| \left[\frac{\sqrt{x}\, L_n^{(\alpha+1)}(x)}{\sqrt{x_{kn}^{(\alpha+1)}}\, L_n^{(\alpha+1)\prime}(x_{kn}^{(\alpha+1)})(x - x_k)} \right]^2,$$
$$(x_k \equiv x_{kn}^{(\alpha+1)}) = A_1 + A_2 \text{ (say)}. \tag{32}$$

$$A_1 = |f(x) - f(0)| \left[\frac{L_n^{(\alpha+1)}(x)}{L_n^{(\alpha+1)}(0)} \right]^2$$
$$= \begin{cases} \omega(f, x), & 0 \leqslant x \leqslant c/n \\ \omega(f, x)\, O[x^{-\alpha/2 - 3/4}\, O(n^{-\alpha/2 - 3/4})]^2, & c/n \leqslant x \leqslant \Omega. \end{cases} \tag{33}$$
$$= \begin{cases} O(\omega_\Delta(1/n)); & 0 \leqslant x \leqslant c/n \\ O(\omega_\Delta(1/\sqrt{n})); & c/n < x \leqslant \Delta + \varepsilon \end{cases}$$

$$A_2 = \sum_{k=1}^{n} |f(x)-f(x_k)|\,|v_k^{(\alpha+1)}(x)|\left[\frac{\sqrt{x}\,L_n^{(\alpha+1)}(x)}{\sqrt{x_k}\,L_n^{(\alpha+1)\prime}(x_k)(x-x_k)}\right]^2;$$

$(x_k = x_{kn}^{(\alpha+1)})$.

but for $x \in J$ and $x_{kn}^{(\alpha+1)} \leqslant \Delta+\varepsilon$ we have

$$A_2 = O\left\{\sum_{k=1}^{n} \omega(f, |x-x_{kn}^{(\alpha+1)}|)\,|v_k^{(\alpha+1)}(x)|\left[\frac{L_n^{(\alpha)}(x)}{(x-x_{kn}^{(\alpha+1)})\,L_n^{(\alpha)\prime}(x_{kn}^{(\alpha)})}\right]^2\right\}. \quad (34)$$

Let $x \notin J$ then we can choose $x^* \in J$ such that $|x-x^*| \leqslant c/n$. Again we have

$$A_2 = O\left(\sum_{k=1}^{n} \omega(f, |x^*-x_{kn}^{(\alpha)}|)\,|v_k^{(\alpha+1)}(x)|\left[\frac{L_n^{(\alpha)}(x^*)}{(x^*-x_{kn}^{(\alpha)})\,L_n^{(\alpha)\prime}(x_{kn}^{(\alpha)})}\right]^2\right).$$

We write $A_i = O\left(\omega\left(f, \frac{ij+i^2}{n}\right)\right)$, so we have

$$A_2 \equiv \sum = \sum_{k=j} + \sum_{\substack{|x-x_k|\leqslant\xi(\alpha)\\ k\neq j}} + \sum_{\substack{|x-x_k|>\xi(\alpha)\\ x_k\leqslant\Delta+\varepsilon}} + \sum_{\substack{|x-x_k|>\xi(\alpha)\\ x_k>\Delta+\varepsilon}} = \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}$$

(a) Let $c/n \leqslant x \leqslant \Delta+\varepsilon$ so that by

$$\frac{L_n^{(\alpha)}(x)-L_n^{(\alpha)}(x_j)}{x-x_j} = -L_n^{(\alpha)\prime}(x^*);\ x^* \in [x, x_j]$$

$$\mathrm{I} = O(\omega(f, |x-x_j|)) = O(\omega_\Delta(1/n), \quad (35)$$

Now $|x-x_k| \leqslant \xi(\alpha) \Rightarrow k \sim \sqrt{n}$ and by (20) $|x-x_k| \sim k^2/n$ and (21), (22) and (28) yield

$$\begin{aligned}
\mathrm{II} &= O\left(\sum_{i=1}^{[\sqrt{n}]} A_i k^{2\alpha+3} n^{-2\alpha-2} \frac{n^2}{k^4} n^{\alpha-1/2} x^{-\alpha-1/2}\right)\\
&= O(x^{-\alpha-1/2} n^{-\alpha-1/2}) \sum_{i=1}^{[\sqrt{n}]} A_i i^{2\alpha-1},\ x \in [c/n, \Delta+\varepsilon],\ k \sim i,\\
&= O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n})\, i^{2\alpha-1}\right);\ -1/2 \leqslant \alpha < 1,\\
&= O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n})\, i^{-2}\right);\ -1 < \alpha < -1/2.
\end{aligned} \quad (36)$$

$$\begin{aligned}
\mathrm{III} &= O\left(\sum_{i=1}^{[\sqrt{n}]} A_i\,|\,v_k^{(\alpha)}(x)|\,k^{2\alpha+3} n^{-2\alpha-2} n^{\alpha-1/2} x^{-\alpha-1/2} \xi^{-2}\right);\\
&= O\left(\sum_{i=1}^{[\sqrt{n}]} A_i k^{2\alpha+3} n^{-2} \frac{n}{k^2}\right); \quad \left(v_k^{(\alpha)}(x) = O(x_k^{-1})\right.\\
&= O\left(\frac{n}{k^2}\right) \text{ for } x_k \leqslant \Delta+\varepsilon\Big); \quad O\left(\sum_{i=1}^{[\sqrt{n}]} A_i k^{2\alpha+1} n^{-1}\right);
\end{aligned}$$

$$= O\left\{\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n})\, k^{2\alpha-1}\frac{k^2}{n}\right\},\ (k \leqslant \sqrt{n});$$

$$= O\left\{\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n})\, k^{2\alpha-1}\right\} = O\left\{\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n})\, i^{2\alpha-1}\right\},\ (k \sim i). \tag{37}$$

We have $f(x) = O(1)$ and $f(x_k) = O(x_k^m)$, so that

$$\text{IV} = \sum_{k=1}^{n} O(x_k^m)\frac{|x_k - (x_k-\alpha-1)(x-x_k)|}{x_k}\frac{x}{x_k}\left[\frac{L_n^{(\alpha+1)}(x)}{L_n^{(\alpha+1)\prime}(x_k)}\right]^2 \xi^{-2},$$

$$= \sum_{k=1}^{n} O(x_k^m)\, x_k^{-2}\, x\,(x^{-\alpha/2-1/2-1/4} n^{\alpha/2+1/2-1/4})^2 [\dot{L}_n^{(\alpha+1)\prime}(x_k)]^{-2},$$

$$= x^{-\alpha-1/2} O(n^{\alpha+1/2}) \sum_{k=1}^{n} x_k^{m-2} [L_n^{(\alpha+1)\prime}(x_k)]^{-2},$$

$$= O(nx^{-1})^{\alpha+1/2} \sum_{k=1}^{n} x_k^{m-1} [L_k^{(\alpha+1)\prime}(x_k)]^{-2};$$

$$\left(x_k > \Delta+\varepsilon \Rightarrow x_k^{-1} < \frac{1}{\Delta+\varepsilon}\right),$$

$$= \begin{cases} O(n^\alpha),\ x \in [c/n, \Delta+\varepsilon],\ \alpha \geqslant -1/2, \\ O(n^{-1/2}),\ x \in [c/n, \Delta+\varepsilon],\ \alpha < -1/2, \end{cases}$$

$$= \begin{cases} O\left(\sum_{[c\sqrt{n}]}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n})\, i^{2\alpha-1}\right);\ \alpha \geqslant -1/2,\ 0 < c < 1, \\ O\left(\sum_{[c\sqrt{n}]}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n})\, i^{-2}\right);\ \alpha < -1/2,\ 0 < c < 1. \end{cases}$$

Thus by (32) and I, II, III and IV we have for $c/n \leqslant x \leqslant \Delta+\varepsilon$,

$$|A_n^{(\alpha)}(f,x) - f(x)| = O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n})\, i^{2\alpha-3}\right),\ -1/2 \leqslant \alpha < 1 \tag{38}$$

$$= O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n})\, i^{-2}\right),\ -1 < \alpha < -1/2;$$

by substituting α in place of $\alpha+1$.

(b) Let $0 \leqslant x \leqslant c/n$ we have

$$|f(0) - A_n^{(\alpha+1)}(f,0)| = 0,$$

and

$$\text{I} = O(1)\,\omega_\Delta(1/n).$$

$$\text{II} = O\left(\sum_{i=1}^{[\sqrt{n}]} \omega(f, |x-x_k|)\, |v_k^{(\alpha)}(x)|\, k^{2\alpha+3} n^{-2\alpha-2} \frac{n^2}{k^4} n^{2\alpha}\right).$$

$$= O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n})\, i^{2\alpha-1}\right),\ \text{(by (28) and } i \sim k).$$

$$\text{III} = O\left(\sum_{i=1}^{[\sqrt{n}]} \omega(f, |x-x_k|\ v_k^{(\alpha)}(x)|\, k^{2\alpha+3}\, n^{-2\alpha-2}\, \xi^{-2}\, n^{2\alpha}\right),$$

$$= O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n})\, x_k^{-1}\, k^{2\alpha+3}\, n^{-2}\right),$$

$$= O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n})\, k^{2\alpha+1}\, n^{-1}\right),$$

$$= O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n})\, k^{2\alpha-1}\, (n^{-1}k^2)\right),$$

$$= O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n})\, i^{2\alpha-1}\right).$$

$$\text{IV} = O(n^{2\alpha}) x \sum_{k=1}^{n} x_k^{m-1} [L_n^{(\alpha+1)\prime}(x_k)]^{-2}$$

$$= O(n^{\alpha-1}) x = O(n^{\alpha-2})$$

$$= O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n})\, i^{2\alpha-1}\right),\ \alpha > -1,$$

So in $[0, c/n]$ we too get

$$|f(x) - A_n^{(\alpha)}(f, x)| = O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n})\, i^{2\alpha-3}\right).$$

Thus in the sense of uniform norm,

$$\| A_n^{(\alpha)}(f, x) - f(x) \|_{[0,\Delta]} = \begin{cases} O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n})\, i^{2\alpha-3}\right), & -1/2 \leqslant \alpha < 1 \\ O\left(\sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}(i^{-2}\right) - 1 < \alpha < -1/2. \text{ Q.E.D.} \end{cases}$$

4. We use theorem 4.1 of Vertesi [5] for the proof of theorem 2, which is as follows:

4.1. *Theorem* 4.1 *of Vertesi.*

If for the sequence of linear operators $T_n(f, x)$ and the functions $g_n(x)$, $(n = 1, 2, \ldots)$ we have

(a) $g_n(x) \in C(\omega)$,

(b) $T_n(g_n, z_n) \geqslant c_1 \lambda_n(z_n)$ for certain $\{z_n\} C[-1, 1]$, (definition of $\lambda_n(z_n)$).

(c) $\overline{f}(x) \overset{\text{def}}{=} Q \sum_{i=1}^{\infty} e_{n_i} g_{n_i}(x) \in C(\omega)$ for certain $\{n_i\}$ and $\{e_{n_i}\}$, $(0 < e_n \leqslant 1,\ e_{n+1} \leqslant e_n)$;

(d) $c_2 \lambda_{n_k}(z_{n_k}) > \sum_{i=k+1}^{\infty} e_{n_i} |T_{n_k}(g_{n_i}; z_{n_k})| + \sum_{i=k}^{\infty} e_{n_i} |g_{n_i}(z_{n_k})|$, $(k = 1, 2 \ldots)$;

with $c_2 < c_1$ then there exists an $f \in C(\omega)$ such that

$$T_n(f, z_n) - f(z_n) > e_n \lambda_n(z_n), \ (n = n_1, n_2 \ldots).$$

4.2 *Proof of theorem* 2.

Let

$$c_\Delta(w) = \{g_n(x): g_n(x) \text{ are continuous for } 0 \leqslant x \leqslant \Delta \text{ and } g_n(x) \equiv 0 \text{ for } x > \Delta,$$
$$\omega(g_n, t) = O(\omega(t)); \lim_{t \to 0} \omega(t) t^{-1} = \infty\}.$$

We select a subset of $C_\Delta(\omega)$ as follows

$$g_n(z_n) = g_n(0) = 0, \ 0 \leqslant z_n \leqslant x_1, \ (x_{1n}^{(\alpha)} = x_1),$$
$$g_n(z_n) = \omega(k/\sqrt{n}) \text{ sign } v_k^{(\alpha+1)}(x); \ x_1 \leqslant z_n \leqslant x_k \leqslant \Delta,$$
$$g_n(z_n) = g_n(x_k) = 0, \ x_k \geqslant \Delta.$$

Thus

$$T_n(g_n, z_n) \equiv A_n^{(\alpha+1)}(g_n, z_n) = \sum_{k=0}^{n} g_n(x_k) a_{kn}^{(\alpha+1)}(z_n),$$

where

$$a_{on}^{(\alpha+1)}(z_n) = \left[\frac{L_n^{(\alpha+1)}(z_n)}{L_n^{(\alpha+1)}(0)}\right]^2, \quad a_{kn}^{(\alpha+1)}(z_n)$$

$$= v_k^{(\alpha+1)}(z_n)\left[\frac{\sqrt{z_n}\, L_n^{(\alpha+1)}(z_n)}{\sqrt{x_k}\, L_n^{(\alpha+1)\prime}(x_k)(z_n - x_k)}\right]^2$$

so that

$$A_n^{(\alpha+1)}(g_n, z_n) = \sum_{k=0}^{n} g_n(x_k) a_{kn}^{(\alpha+1)}(z_n),$$

$$\geqslant c_4 \sum_{x_k \leqslant \Delta} \omega(k/\sqrt{n})\left[\frac{\sqrt{z_n}\, L_n^{(\alpha+1)}(z_n)}{\sqrt{x_k}\, L_n^{(\alpha+1)\prime}(x_k)(z_n - x_k)}\right]^2$$

$$\overset{\text{def}}{=} \lambda_n(z_n) \sim \left[c_s \sum_{|x - x_k| < \varepsilon} \omega(k/\sqrt{n}) \frac{l_k^2(z_n)}{x_k} + c_6 \sum_{\substack{|x - x_k| \geqslant \varepsilon \\ x_k \leqslant \Delta}} \omega(k/\sqrt{n}) \frac{l_k^2(z_n)}{x_k^2}\right];$$

$$\left(l_k(z_n) = \frac{L_n^{(\alpha)}(z_n)}{L_n^{(\alpha)\prime}(x_k)(z_n - x_k)}\right);$$

$$\sim \left[c_7 \sum_{i=1}^{[\sqrt{n}]} \omega(k/\sqrt{n}) i^{2\alpha-1} + n^{-1} c_8 \sum_{i=1}^{[\sqrt{n}]} \omega(k/\sqrt{n}) i^{2\alpha-1}\right]$$

$$\sim c_g \sum_{i=1}^{[\sqrt{n}]} \omega(i/\sqrt{n}) i^{2\alpha-1}; \ (-\tfrac{1}{2} \leqslant \alpha < 1).$$

We, therefore, conclude that for certain $f \in C(m)$, $-1/2 \leqslant \alpha < 1$, $\{z_n\} c [0, \Delta]$,

$$\|A_n^{(\alpha+1)}(f, z_n) - f(z_n)\|_{[0,\Delta]} > \sum_{i=1}^{[\sqrt{n}]} \omega_\Delta(i/\sqrt{n}) i^{2\alpha-1}$$

or

$$\|A_n^{(\alpha)}(f, z_{n_1}) - f(z_{n_1})\|_{[0,\Delta]} > \sum_{i=1}^{[\sqrt{n}]} \omega_\Delta (i/\sqrt{n})\, i^{2\alpha-3}.$$

Similar arguments hold for other inequality. Q.E.D.

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References

[1] Szabados J, On the covergence of Hermite-Fezér interpolation for the Laguerre Abscissas, *Acta. Math. Acad. Sci. Hungarica* **24** (1973) 243–250

[2] Szász P, The extended Hermite-Fezér interpolation formula with application to the theory of generalized almost-step parabolas, *Publ. Maths Debrecen* **11** (1964) 85–100

[3] Szegö, G, Orthogonal polynomials *Am. Math. Soc.* Colloq. Publ. (3rd ed.) New York (1967)

[4] Vertesi P, Hermite Fezér interpolation based on the roots of Laguerre polynomials. *Stud. Sci. Math. Hungarica* (1971) 91–97

[5] Vertesi P, Hermite Fezér type of interpolation I. *Acta Math. Acad. Sci. Hungarica* **32** (1978) 349–369

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Quotient-bounded elements in locally convex algebras. II

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Abstract. Consideration of quotient-bounded elements in a locally convex GB^*-algebra leads to the study of proper GB^*-algebras viz those that admit nontrivial quotient-bounded elements. The construction and structure of such algebras are discussed. A representation theorem for a proper GB^*-algebra representing it as an algebra of unbounded Hilbert space operators is obtained in a form that unifies the well-known Gelfand–Naimark representation theorem for C^*-algebra and two other representation theorems for b^*-algebras (also called LMC*-algebras), one representing a b^*-algebra as an algebra of quotient bounded operators and the other as a weakly unbounded operator algebra. A number of examples are discussed to illustrate quotient-bounded operators. An algebra of unbounded operators constructed out of noncommutative L^p-spaces on a regular probability gauge space and the convolution algebra of periodic distributions are analyzed in detail; whereas unbounded Hilbert algebras and L^∞-integral of a measurable field of C^*-algebras are discussed briefly.

Keywords. Generalized B*-algebras; unbounded representations; quotient-bounded elements: universally bounded elements; unbounded Hilbert algebras; locally multiplicative convex (LMC) algebras.

1. Introduction

The following two generalizations of abstract C^*-algebras at the level of topological algebras have been investigated in the literature.

1.1. *Definition* ([3], [19], [23])

A *b^*-algebra* (also called an LMC*-algebra) is a complete topological algebra A with a continuous involution $x \to x^*$ such that

(i) A is a locally m-convex algebra i.e. its topology is determined by a separating family $P = (p_\alpha)$ of sub-multiplicative seminorms, and

(ii) for each α, $p_\alpha(x^*x) = p_\alpha(x)^2$ for all $x \in A$.

A commutative b^*-algebra A with 1 admits a faithful representation as the algebra of all continuous complex valued functions on a completely regular space [23], whereas in the noncommutative case, there are two such representation theorems viz one due to Inoue [19] (see also [12]) representing A faithfully as a weakly unbounded algebra of unbounded Hilbert space operators, and the other due to Giles *et al* [17] wherein A is shown to be isomorphic to an algebra of quotient-bounded operators on a locally convex space.

1.2 *Definition* ([2], [14])

Let A be a locally convex algebra with 1 and with a continuous involution. An element $x \in A$ is called (*Allan*) *bounded* if for some $\lambda \neq 0$, $\{(\lambda^{-1}x)^n | n = 1, 2, \ldots\}$ is bounded. The algebra A is called *symmetric* if for each $x \in A$, $(1 + x^*x)^{-1}$ exists and is bounded. Let $\mathscr{B}^*$ be the collection of all $B \subset A$ such that B is absolutely convex, $B^2 \subset B$, $B^* = B$ and B is closed and bounded. A locally convex* algebra with 1 is called a *Generalized B*-algebra* (*GB*-algebra*) if

(i) A is symmetric,

(ii) the collection $\mathscr{B}^*$ has a greatest member B_0 under inclusion, called the unit ball of A; and

(iii) the *subalgebra $A(B_0) = \{\lambda x | \lambda \in \mathbb{C}, x \in B_0\}$ is a Banach algebra with the Minkowski functional $\|\cdot\|_{B_0}$. (It turns out to be a C^*-algebra.)

Allan [2] has shown that such an algebra A, if commutative, is isomorphic to an algebra of continuous functions on a compact Hausdorff space taking values in the extended complex place $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ such that $f^{-1}(\infty)$ is atmost a nowhere dense set. Dixon [14] has shown that a (locally convex) GB^*-algebra admits a faithful representation as a certain algebra of closed operators (called an *extended C*-algebra*) all defined on a common dense subspace of a Hilbert space.

Now as shown in [2], a b^*-algebra is a GB^*-algebra. Thus it is natural to synthesize and unify the above Gelfand–Naimark type representation theorems. Here we present GB^*-representation theorems in forms that quickly reduces to the b^*-theorems mentioned above when the given algebra is a b^*-algebra. This we do for a class of GB^*-algebras that arise naturally in the study of quotient-bounded and universally-bounded elements introduced in [9].

1.3. *Definition*

Let A be a locally convex algebra with 1 (and with a separately continuous multiplication). Let $P(A)$ denote the collection of all calibrations P on A viz families P of seminorms determining the topology of A. Let $P = (p_\alpha | \alpha \in \Delta)$ be in $P(A)$. An element $a \in A$ is called (*left*) *P-quotient bounded* if for each α, there is a real constant $M_{\alpha,a}$ depending on α and a such that $p_\alpha(ax) \leqslant M_{\alpha,a}\, p_\alpha(x)$ holds for all $x \in A$. Further, it is called (*left*) *P-universally bounded* if the $M_{\alpha,a}$, for all α, have an upper bound depending only on a (written M_a).

Note that in a calibrated hypocontinuous locally convex algebra A, a universally-bounded element is (Allan) bounded; whereas a quotient-bounded element need not be; e.g. let A be an LMC algebra with an m-calibration P. Then each $a \in A$ is P-quotient bounded; on the other hand, $a \in A$ is (Allan) bounded iff a is Q-universally bounded for some m-calibration Q on A.

Given a locally convex algebra A with a calibration P, let B_P (respectively Q_P) be the set of all P-universally bounded (respectively P-quotient bounded) elements. In [9], it is shown that Q_P is a subalgebra of A and B_P is a subalgebra of Q_P with $1 \in B_P$. A natural locally m-convex (LMC) topology t_P is defined on Q_P by the m-calibration $\{q_\alpha | \alpha \in \Delta\}$ where $q_\alpha(a) = \sup\{p_\alpha(ax) | p_\alpha(x) \leqslant 1\}$ $(a \in Q_P)$, whereas $p(a) = \sup_\alpha q_\alpha(a)$ defines an

algebra norm on B_P. If A is complete, then each of (B_P, p) and (Q_P, t_P) is complete. Further, if A is a hypocontinuous locally convex GB^*-algebra with unit ball B_0, and P is a GB^*-calibration (so that A admits a P-hermitian decomposition [32, Theorem 8.15]), then

(i) (Q_P, t_P) is a b^*-algebra having $\{q_\alpha\}$ as a b^*-calibration, and

(ii) (B_P, p) is a C^*-algebra with $B_P = \{\lambda x \mid \lambda \in \mathbb{C},\ x \in B_0\} = A(B_0)$ and $p(x) = \inf\{\lambda > 0 \mid x \in \lambda B_0\} = \|x\|_{B_0}$.

Thus when A is a b^*-algebra with a b^*-calibration P, then $Q_P = A$. However, in general Q_P may be trivial. In fact, in [9], we have an example in which there is no b^*-algebra properly lying between $A(B_0)$ and A. This motivates the following.

1.4 *Definition*

A locally convex GB^*-algebra A is called *proper* if there is a GB^*-calibration P on A (called a *proper GB^*-calibration*) such that Q_P is nontrivial in the sense that $B_P \neq Q_P \neq A$. It is called *pure* if $A(B_0) \neq A$.

In §2, we discuss a certain method of constructing proper GB^*-algebras and a decomposition theorem for such algebras. The functional representation theorem [14, Theorem 4.6] and the operator representation theorem [14, Theorem 7.1] are suitably modified for proper GB^*-algebras in such a way that in the latter case, the quotient-bounded elements of A are mapped to quotient-bounded operators on a locally convex space [17]. Our theorem can be regarded as a GB^*-analogue of a representation theorem for b^*-algebras [17, Theorem 8]. In §3, we consider a couple of examples to illustrate the concepts. After briefly mentioning Arens algebra L^ω on a finite measure space and its σ-finite variant, we discuss in detail its now commutative analogue on a regular probability gauge space. We also have the Hardy–Arens algebra H^ω on the disc and its noncommutative analogue determined by a one-parameter group of *automorphisms on a van Newmann algebra. Q_p and B_p are also computed for unbounded Hilbert algebras recently studied by Inoue [21]. We also introduce L^p- and L^ω-direct integrals of a measurable field of C^*-algebras. The construction of these examples is analogous. Finally we analyze in detail the convolution algebra of periodic distributions on the circle.

In the sequel, we shall frequently need the following ([4], [5]).

1.5 *Theorem*

(a) Let A be a locally convex GB^*-algebra with unit ball B_0. Then for each $x \in A$, $x_n = x([1 + 1/n]x^*x)^{-1}$ is in $A(B_0)$ for $n = 1, 2, \ldots$ and $x_n \to x$.

(b) Let A be a complete hypocontinuous locally convex *algebra with 1. If there exists a *subalgebra B of A containing 1 such that

(i) B is a B^*-algebra under some norm $\|\cdot\|$, and

(ii) the inclusion $B \to A$ is a continuous injection (with $\|\cdot\|$ on B) with sequentially dense range, then A is a GB^*-algebra.

2. Main results

2.1 *Proposition*

Let (A_α) be an inverse system of an infinite family of pure complete hypocontinuous locally convex GB^*-algebras. Then the following hold.

(i) The inverse limit $A = \overset{\lim}{\leftarrow} A_\alpha$ is a pure complete hypocontinuous locally convex GB^*-algebra.

(ii) For each α, let P_α be a GB^*-calibration on A_α. Let P be the natural calibration on A. Then P is a GB^*-calibration on A. If $B_{P^\alpha} = Q_{P^\alpha}$ for each α and if A contains a P-quotient-bounded hermitian element that is not (Allan) bounded, then P is a proper GB^*-calibration.

In the course of the proof, we compute Q_P and B_P for A. It also follows that the product of an infinite family of pure complete hypocontinuous GB^*-algebras A_α admitting improper GB^*-calibrations (i.e. $Q_{P^\alpha} = B_{P^\alpha}$) is a proper GB^*-algebra.

Proof. Let $\{(A_\alpha, t_\alpha) | \alpha \in \Delta\}$ be an infinite family of complete hypocontinuous pure locally convex GB^*-algebras. Let B_α be the unit ball of A_α and 1_α the identity of A_α. Let $B = \prod_\alpha A_\alpha$ be the cartesian product of A_α's with the product topology t. Then it is a routine verification that (B, t) is a complete locally convex GB^*-algebra with unit ball $B_0' = \Pi B_\alpha$.

A simple argument also shows that the product of hypocontinuous locally convex algebras is hypocontinuous.

Now assume that Δ is directed. For $\alpha \leqslant \beta$ in Δ, let $\pi_{\alpha\beta}$: $(A_\beta, t_\beta) \to (A_\alpha, t_\alpha)$ be onto continuous* algebra homomorphisms; and for each α, let π_α: $B \to A_\alpha$ be the natural projection. Suppose that A_α's with the maps $\pi_{\alpha\beta}$ form an inverse system. Let $(A, t) = \overset{\lim}{\leftarrow}(A_\alpha, t_\alpha) = \{x = (x_\alpha) \in \Pi A_\alpha | x_\alpha \in A_\alpha$ for all α and for $\alpha \leqslant \beta$, $\pi_{\alpha\beta}(x_\beta) = x_\alpha\}$ be the inverse limit. Then A, being closed in B, is a complete hypocontinuous locally convex GB^*-algebra with unit ball $B_0 = A \cap B_0'$ so that the C^*-algebra that underlies A is $(A(B_0), \|\cdot\|_{B_0})$ which is a norm closed *subalgebra of $A(B_0') = \{\lambda x | \lambda \in \mathbb{C}, x \in B_0'\}$. The C^*-norm on $A(B_0')$ is $\|x\|_{B_0} = \inf\{\lambda > 0 | x \in \lambda B_0'\}$. Further for $\alpha \leqslant \beta$, $\pi_{\alpha\beta}(B_\beta) \subset B_\alpha$, and so $\pi_{\alpha\beta}(A(B_\beta)) \subset A(B_\alpha)$, $\|\pi_{\alpha\beta}(z)\|_\alpha \leqslant \|z\|_\beta$ for all $z \in A(B_\beta)$. It is easily seen that with $\pi_{\alpha\beta} | A(B_\beta)$, $\alpha \leqslant \beta$ in Δ as morphisms, $\{(A(B_\alpha), \|\cdot\|_\alpha) | \alpha \in \Delta\}$ forms an inverse system of C^*-algebras. Let (Q, τ) be the b^*-algebra obtained by taking the inverse limit of $(A(B_\alpha), \|\cdot\|_\alpha)$'s. Since each $A_\alpha \neq A(B_\alpha)$, $A \neq Q$. Also, the unit ball of (Q, τ), as a GB^*-algebra, is B_0; hence $A(B_0) = \{x \in Q | \sup \|\pi_\alpha(x)\|_\alpha < \infty\}$.

Now to prove (ii), for each α, let $P^\alpha = \{p_\gamma^\alpha | \gamma \in \Delta_\alpha\}$ (Δ_α some index set) be a GB^*-calibration for A_α. Hence by [32, Theorem 8.9], $(A(B_\alpha), \|\cdot\|_\alpha) = (B_{P^\alpha}, p^\alpha)$ viz the C^*-algebra of all P^α universally bounded elements of A_α. The natural calibration on A induced by P^α's is

$$P = \{p_\gamma^\alpha \circ \pi_\alpha | \gamma \in \Delta_\alpha, \alpha \in \Delta\}.$$

Claims: (a) $A(B_0) = B_P$, the P-universally bounded elements of A.

(b) $Q = Q_P$, the P-quotient bounded element of A.

(a) Given $a \in A(B_0)$, $\sup_\alpha \|a_\alpha\|_\alpha = \sup \|\pi_\alpha(a)\|_\alpha < \infty$. Then for each $\alpha \in \Delta$, each $\gamma \in \Delta_\alpha$ and for each $x \in A$, $(p_\gamma^\alpha \circ \pi_\alpha)(ax) \leqslant (\sup_\alpha \|a_\alpha\|_\alpha)(p_\gamma^\alpha \circ \pi_\alpha)(x)$. Hence $a \in B_P$, $A(B_0) \subset B_P$. On the other hand, given $a \in B_P$, there is a constant M_a depending only on a such that $(p_\gamma^\alpha \circ \pi_\alpha)(ax) \leqslant M_a(p_\gamma^\alpha \circ \pi_\alpha)(x)$ for $\alpha \in \Delta, \gamma \in \Delta_\alpha, x \in A$. Hence $a_\alpha = \pi_\alpha(a) \in A(B_\alpha) = B_{P^\alpha}$ and $p^\alpha(a_\alpha) = \sup\{p_\gamma^\alpha(a_\alpha z) | p_\gamma^\alpha(z) \leqslant 1$ for all γ, for $z \in A_\alpha\} \leqslant M_a$. (Here p^α is the natural Banach algebra norm on B_{P^α}, the P^α-universally-bounded elements of A_α). Thus $\sup_\alpha p^\alpha(a_\alpha) < \infty$ and so $a \in A(B_0)$. Thus $B_P \subset A(B_0)$. (This also follows from the fact that B_0 is the greatest member of $\mathscr{B}^*(A)$.)

Note that in view of [32], this implies that P is a GB^*-calibration for A. (b) That $Q \subset Q_P$ follows as in (a). Let $a \in Q_P$. Then for each $\alpha \in \Delta$, $\gamma \in \Delta_\alpha$, $p_\gamma^\alpha(a_\alpha x_\alpha) \leqslant M_{\gamma,a}^\alpha p_\gamma^\alpha(x_\alpha)$ for all $x_\alpha \in A_\alpha$, $x_\alpha = \pi_\alpha(x)$ for $x \in A$, $a_\alpha = \pi_\alpha(a)$. Hence $a_\alpha \in Q_{P^\alpha}$, P^α-quotient bounded elements of A_α. By hypothesis, $a_\alpha \in B_{P^\alpha} = A(B_\alpha)$, P^α being a GB^*-calibration. Thus $a_\alpha \in A(B_\alpha)$ for each α, and so $a \in Q$. Thus $Q = Q_P$.

Now each (Allan) bounded hermitian element of A is in $A(B_0)$ by [14, Lemma 2.6]; with the result, the stated condition implies that $B_P \neq Q_P$. This completes the proof.

In view of [3, Theorem 2.4] and [23, Theorem 5.1], it is immediate that a b^*-algebra can be expressed as an inverse limit of a family of C^*-algebras. This motivates our next result. We say that an inverse system $\{(A_\alpha, t_\alpha) | \alpha \in \Delta\}$ of calibrated locally convex GB^*-algebras with calibration P_α on A_α satisfies *property* (P) provided given x in $\varprojlim A_\alpha$ with associated thread $x \sim (x_\alpha)$, if x_α is P_α-quotient bounded in A_α for each α, then x_α is (Allan) bounded in A_α for each α.

2.2 *Theorem*

A complete metrizable proper locally convex GB^*-algebra is isomorphic to an inverse limit of a family satisfying (P) of pure metrizable calibrated complete locally convex GB^*-algebras.

Proof. Let (A, t) be a given GB^*-algebra with a proper GB^*-calibration $P = (p_\alpha | \alpha = 1, 2, \ldots)$. Let, for each α, $N_\alpha = \{a \in Q_P | q_\alpha(a) = 0\}$ which is a t_P closed* ideal of Q_P. Since (Q_P, t_P) is a b^*-algebra having $\{q_\alpha\}$ as a b^*-calibration, $\tilde{q}_\alpha(a + N_\alpha) = q_\alpha(a)$ $(a \in Q_P)$ defines a C^*-norm on $(Q_P)_\alpha = Q_P/N_\alpha$. By [3, Theorem 2.4], $((Q_P)_\alpha, \tilde{q}_\alpha)$ is complete. As in [23, Theorem 5.1], $(Q_P, t_P) = \varprojlim ((Q_P)_\alpha, \tilde{q}_\alpha)$, where $\{N_\alpha\}$ is directed as $\alpha \leqslant \beta$ if $N_\beta \subset N_\alpha$; and in this case, $\pi_{\alpha\beta}: (Q_P)_\beta \to (Q_P)_\alpha$ is $\pi_{\alpha\beta}(a + N_\beta) = a + N_\alpha (a \in Q_P)$. Let $\pi_\alpha(a) = a + N_\alpha$ be the natural map from Q_P onto $(Q_P)_\alpha$. Let t_q^α be the quotient topology on $(Q_P)_\alpha$ induced from the topology t. Let $A_\alpha = ((Q_P)_\alpha, t_q^\alpha)^\sim$, the completion of $((Q_P)_\alpha, t_q^\alpha)$. We denote the topology on A_α by t_q^α.

It is easily seen that $((Q_P)_\alpha, t_q^\alpha)$ is a locally convex *algebra, and that the algebraic structure on $(Q_P)_\alpha$ can be uniquely extended to A_α making A_α a complete locally convex *algebra with identity. On the other hand, let $\bar{N}_\alpha$ be the closure of N_α in (A, t). Since multiplication is jointly continuous in A, it is easily seen that N_α is an ideal in A; and $A/\bar{N}_\alpha$ with the quotient topology τ induced by t is a complete locally convex *algebra.

Claim. $(A/\bar{N}_\alpha, \tau)$ is homeomorphic to (A_α, t_q^α).

Indeed, let ϕ: $(Q_P)_\alpha \to A/\bar{N}_\alpha$ be $\phi(a+N_\alpha) = a+\bar{N}_\alpha$. Since P is a GB^*-calibration, the Banach algebra (B_P, p) is identical to the C^*-algebra $(A(B_0), \|\cdot\|_{B_0})$ where B_0 is the unit ball of A. From Theorem 1.5(a), it is immediate that the range of ϕ is sequentially dense in $A/\bar{N}_\alpha$. Also, ϕ is one-to-one, and for any $p_\beta \in P$, $a \in Q_P$,

$$\ddot{p}_\beta(a+\bar{N}_\alpha) = \inf\{p_\beta(a+y) \mid y \in \bar{N}_\alpha\}$$
$$\leqslant \inf\{p_\beta(a+y) \mid y \in N_\alpha\} = \dot{p}_\beta(a+N_\alpha).$$

Hence ϕ is $t_q^\alpha - \tau$ continuous. Thus ϕ can be extended uniquely to a continuous one-to-one *homomorphism ϕ: $A_\alpha \to A/\bar{N}_\alpha$. Further, ϕ is onto. For, given $x+\bar{N}_\alpha \in A/\bar{N}_\alpha$, Theorem (1.5) (a) gives a sequence (x_n) in $A(B_0)$ such that $x_n \to x$ in A, $x_n + \bar{N}_\alpha \to x+\bar{N}_\alpha$; and $(x_n + N_\alpha)$ being Cauchy in $(Q_P)_\alpha$, converges to some $y \in A_\alpha$. Then $\phi(y) \overset{t_q^\alpha}{=} x+\bar{N}_\alpha$. Now the open mapping theorem implies that ϕ is a homeomorphism.

The natural quotient map π_α: $A \to A/\bar{N}_\alpha$ is identified with the unique continuous extension of π_α: $Q_P \to A_\alpha$ which is an algebra homomorphism. For $x \in A$, $\pi_\alpha(x)^* = \pi_\alpha(x^*)$ is the continuous involution on A_α which is the extension of the involution on $(Q_P)_\alpha$.

Now let τ_P^q be the quotient topology on $(Q_P)_\alpha$ induced by the b^*-topology τ_P in Q_P. It is determined by the calibration $\{\ddot{q}_\beta\}$ where $\ddot{q}_\beta(a+N_\alpha) = \inf\{q_\beta(a+y) \mid y \in N_\alpha\}$. Then $((Q_P)_\alpha, \tau_P^q$ is a Frechet b^*-algebra; and for $a+N_\alpha$ in $(Q_P)_\alpha$, $\ddot{q}_\beta(a+N_\alpha) \leqslant q_\beta(a)$ for all β and $\ddot{q}_\alpha(a+N_\alpha) = q_\alpha(a) = \tilde{q}_\alpha(a+N_\alpha)$. (This is because for $y \in N_\alpha$, $q_\alpha(a) = q_\alpha(a+y-y) \leqslant q_\alpha(a+y) + q_\alpha(y) = q_\alpha(a+y)$). Thus, by the open mapping theorem, τ_P^q coincides with the norm topology given by $\tilde{q}_\alpha$.

Further, $N_\alpha \cap A(B_0)$ is a norm closed *ideal of the C^*-algebra $A(B_0)$. Consider the quotient C^*-algebra $C = A(B_0)/N_\alpha \cap A(B_0)$ with the quotient norm $\ddot{p}(a+N_\alpha \cap A(B_0)) = \inf\{p(a+y) \mid y \in N_\alpha \cap A(B_0)\}$. (Note that P being a GB^*-calibration on A, $p = \|\cdot\|_{B_0}$ on $A(B_0) = B_P$). Then we can identify the C^*-algebras $(C, \ddot{p})$ and $((Q_P)_\alpha, \tilde{q}_\alpha)$. Indeed, the embedding id : $C \to (Q_P)_\alpha$ is a well-defined isomorphism of C into $(Q_P)_\alpha$. Hence it is an isometry and id(C) is a $\tilde{q}_\alpha$-closed *subalgebra of $(Q_P)_\alpha$. On the other hand, id(C) is dense in $(Q_P)_\alpha$. For, let $a+N_\alpha \in (Q_P)_\alpha$ with $a \in Q_P$. By Theorem 1.5 (a) (applied to Q_P as a GB^*-algebra), we get a sequence (a_n) in $A(B_0)$ such that for each β, $q_\beta(a_n - a) \to 0$. Hence in particular, $(\ddot{q}_\alpha(a_n+N_\alpha) - (a+N_\alpha)) \to 0$. Thus it follows that

$$A(B_0)/N_\alpha \cap A(B_0) \simeq (Q_P)_\alpha \text{ and } \ddot{p} = \tilde{q}_\alpha.$$

From above and Theorem 1.5 (b), it follows that (A_α, t_q^α) is a complete metrizable locally convex GB^*-algebra with the underlying C^*-algebra $((Q_P)_\alpha, \tilde{q}_\alpha)$.

Now for $\alpha \leqslant \beta$, consider $\pi_{\alpha\beta}$: $((Q_P)_\beta, t_q^\beta) \to ((Q_P)_\alpha, t_q^\alpha)$. Since for each $\delta \in \mathbf{N}$, $\inf\{p_\delta(a+y) \mid y \in N_\alpha\} \leqslant \inf\{p_\delta(x+y) \mid y \in N_\beta\}$, $\pi_{\alpha\beta}$ is a continuous homomorphism, and in view of Theorem 1.5(a), extends to a continuous surjective *homomorphism $\tilde{\pi}_{\alpha\beta}$: $(A_\beta, t_q^\beta) \to (A_\alpha, t_q^\alpha)$. Given $x \in A$, if x_n is as in Theorem 1.5(a), then $\pi_\beta(x_n) = \pi_\beta(x)(1+(1/n)\pi_\beta(x)^*\pi_\beta(x))^{-1} \to \pi_\beta(x)$ and so $\pi_{\alpha\beta}(\pi_\beta(x) = \pi_\alpha(x)$. Thus $\{(A_\alpha, t_q^\alpha), \pi_{\alpha\beta} \mid \alpha \leqslant \beta\}$ forms an inverse system. Let $B = \varprojlim (A_\alpha, t_q^\alpha)$ be the inverse limit. Define π: $A \to B$ as $\pi(x) = (\pi_\alpha(x))$. Then π is a topological *isomorphism of A onto B.

Finally, identifying A and B, let $a = (a_\alpha)$ be such that for each α, $a_\alpha = a + \bar{N}_\alpha$ is P^α-quotient bounded in A^α for all each α, where P^α is the natural quotient calibration on A^α for t_q^α viz $P^\alpha = \{\ddot{p}_\beta \mid \beta \in \mathbb{N}\}$ where $\ddot{p}_\beta(x + \bar{N}_\alpha) = \inf\{p_\beta(a + y) \mid y \in \bar{N}_\alpha\}$. Then for each fixed α, there is a constant $M_{\beta,a}$ such that

$$\ddot{p}_\beta((a + \bar{N}_\alpha)(x + \bar{N}_\alpha)) \leqslant M_{\beta,a}\ddot{p}_\beta(x + \bar{N}_\alpha)$$
$$\leqslant M_{\beta,a}p_\beta(x) \text{ for all } x \in A.$$

In particular, $\ddot{p}_\alpha(ax + \bar{N}_\alpha) \leqslant M_{\alpha,a}p_\alpha(x)$ for all $x \in A$. Now for each $y \in \bar{N}_\alpha$, $p_\alpha(ax) = p_\alpha(ax + y - y) \leqslant p_\alpha(ax + y) + p_\alpha(y) = p_\alpha(ax + y)$ since $\bar{N}_\alpha \subset \ker p_\alpha$. Thus $p_\alpha(ax) \leqslant M_{\alpha,a}p_\alpha(x)$ for all $x \in A$. Hence $a \in Q_P$, and so $a_\alpha \in (Q_P)_\alpha$ for each α. Thus a_α is (Allan) bounded in A_α for each α. This completes the proof.

A *algebra of continuous functions [14, Definition 4.7] on a topological space M is a collection F of extended complex valued continuous functions on M such that each $f \in F$, $f^{-1}(\infty)$ is at most a set of first category and F is a *algebra under operations $f + y$, $\lambda f (\lambda \in \mathbb{C})$, fg which consists in defining these operations pointwise on the dense set where values involved are finite and then extending the resulting functions to be continuous extended complex valued functions on M. Let A be a commutative GB^*-algebra with unit ball B_0. By a theorem due to Allan [14, Theorem 4.6], the Gelfand representation of $A(B_0)$ onto $C(X)$ (where X is the compact Hausdorff maximal ideal space of $A(B_0)$) extends uniquely to a *isomorphism of A onto a *algebra of extended complex valued continuous functions on X. On the otherhand, Michael [23] and Apostol [3] have discussed the realization of a commutative b^*-algebra as the algebra of all continuous complex-valued functions on a completely regular space. We synthesize both these results in the following.

2.3 *Theorem*

Let A be a proper commutative locally convex GB^*-algebra with a proper GB^*-calibration P. Then there exists a real-compact completely regular Hausdorff space Z and a mapping π of A into extended complex-valued continuous functions on Z such that

(a) π is a *isomorphism of A onto a *algebra of functions on Z containing $C(Z)$, the algebra of all continuous complex-valued functions on Z;
(b) π represents B_P *isomorphically onto $C_b(Z)$, the C^*-algebra of all bounded continuous complex-valued functions on Z; and
(c) π represents Q_P*isomorphically onto $C(Z)$.

Proof. Let X (respectively Z) be the usual weakly topological space of all non-zero continuous multiplicative functionals (respectively all non-zero multiplicative functionals) on Q_P. Then as in [3, Corollary 3.4], X is a completely regular Hausdorff space and $X \subsetneq Z = \nu X \subsetneq M = \beta X$ are dense continuous embeddings. Here νX and βX are respectively the real-compactification and the Stone–Cech compactification of X, and M is the maximal ideal space of B_P. By results in [3], we have the algebraic *isomorphisms $Q_P \simeq C(X) \simeq C(Z)$ and $B_P \simeq C_b(X) \simeq C_b(Z) \simeq C(M)$ defined as $x \to \tilde{x}\,(x \in Q_P)$ where $\tilde{x}(\varphi) = \varphi(x)$ $(\varphi \in Z)$. If $x \to \hat{x}$ is the Gelfand representation of B_P

onto $C(M)$, then $\hat{x}|_Z = \tilde{x} = \phi(x)$ (say). By [14, Theorem 4.6], ϕ extends uniquely to a *isomorphism of A onto a *algebra of functions on M defined as $\hat{x}(\varphi) = \varphi'(x)$ where φ' is the unique $\not{c}^* = \not{c} \cup \{\infty\}$ valued 'partial homomorphism' that extends φ as in [2, Proposition 3.1]. Hence for each $a \in Q$, a agrees with the unique Stone extension [18, §6.5] of $\tilde{a} \in C(Z) \simeq C(X)$ as a $\not{c}^*$-valued continuous function on M. It follows that $\hat{a}|_Z = \tilde{a}$ $(a \in Q_P)$. The map ϕ on A defined as $\phi(a) = \hat{a}|_Z$ is the desired map.

Now we consider operator representations. A **algebra of closed operators* [14, Definition 7.1] on a Hilbert space H is a set $\mathscr{A}$ of closed operators in H that forms a *algebra under the operations $T, S \to (T+S)^-$, $(T, S) \to (TS)^-$ and $T \to T^*$. (Here $^-$ denotes the closure of the operator and * is the operator adjoint). It is called an *extended C^*-algebra* if $\mathscr{A} \cap \mathscr{B}(H)$ is a C^*-algebra and for each $T \in \mathscr{A}$, $(1 + T^*T)^{-1} \in \mathscr{A}$. ($\mathscr{B}(H)$ is the C^*-algebra of all bounded linear operators on H). It is said to have *a common dense domain* D if $D = \cap \{D(T) | T \in \mathscr{A}\}$ is dense in H where $D(T)$ denotes the domain of T. Such an algebra is an *EC^*-algebra* in the sense of Inoue [21].

In [20], Inoue introduced weakly unbounded operator algebras. Let $\{\mathscr{A}_\alpha | \alpha \in \Delta\}$ be a family of C^*-algebras, $\mathscr{A}_\alpha$ acting on a Hilbert space H_α. Let $H = \Sigma^{\oplus} H_\alpha$ be the Hilbert space direct sum and let $D = \{\xi = (\xi_\alpha) \in H | \xi_\alpha \in H_\alpha$ for all α, $\xi_\alpha = 0$ for all but finitely many α's$\}$. It is dense in H. A *algebra $\mathscr{A}$ of closable operators all defined on D is called a *weakly unbounded operator algebra* if $\mathscr{A}$ is a *subalgebra of $\Pi \mathscr{A}_\alpha = \{T = (T_\alpha) | T_\alpha \in \mathscr{A}_\alpha$ for all $\alpha\}$ such that $\mathscr{A}_b = \{\overline{T} | T \in \mathscr{A}$ is bounded$\} = \Sigma \mathscr{A}_\alpha$ the usual (1^∞) direct sum of C^*-algebras. Note that $\mathscr{A}$ is a pre b^*-algebra under a natural topology.

Giles *et al* [17] have introduced quotient-bounded and universally-bounded operators on a locally convex space. Let X be a locally convex (Hausdorff) space with a calibration $\Gamma = (p_\alpha | \alpha \in \Delta)$. A linear operator $T: X \to X$ is called *quotient-bounded* if for each $\alpha \in \Delta$, there is a scalar $k_\alpha \geqslant 0$ such that $p_\alpha(Tx) \leqslant k_\alpha p_\alpha(x)$ $(x \in A)$. It is called *universally bounded* if these k_α's for all α have an upper bounded written k. Let $Q(X, \Gamma)$ (respectively $B(X, \Gamma)$) be the set of all quotient-bounded (respectively universally bounded) operators on X. Then $Q(X, \Gamma)$ forms an algebra and $B(X, \Gamma)$ is a subalgebra of $Q(X, \Gamma)$. The seminorms $q_\alpha(T) = \sup \{p_\alpha(Tx) | p_\alpha(x) \leqslant 1\}$ define a natural LMC topology on $Q(X, \Gamma)$, where as $p_\Gamma(T) = \sup_\alpha q_\alpha(T)$ defines an algebra norm on $B(X, \Gamma)$. If X is complete, then each of these is complete.

Dixon [14, Theorem 7.1] has faithfully represented a locally convex GB^*-algebra A as an extended C^*-algebra. Giles *et al* [17, Theorem 8] have proved that a b^*-algebra is *isomorphic to a *subalgebra of quotient-bounded operators on the product of a family of Hilbert spaces. The following representation theorem for proper GB^*-algebras synthesizes both these results. Note that in the course of the proof, we also show that a b^*-algebra is *isomorphic to a weakly-unbounded operator algebra. For basic notions of unbounded representations, we refer to [24].

2.4 *Theorem*

Let (A, t) be a proper complete bornological locally convex GB^*-algebra with a proper GB^*-calibration P. Then there exists a complete locally convex space X and a subspace

H of X such that H is a Hilbert space that is continuously and densely injected in X; and there exists a representation π of A into linear operators on X such that

(a) π represents A faithfully as an extended C^*-algebra with a common dense domain in H;

(b) π represents Q_P faithfully and continuously as a subalgebra Q of the algebra of quotient-bounded operators on X which, in turn, is isomorphic to a weakly-unbounded operator algebra on H; and

(c) π represents B_P faithfully as a subalgebra B of the algebra of universally bounded operators on X which, in turn, is isomorphic to a C^*-algebra of bounded operators on H.

Proof. For f in $P(A)$, the set of all positive linear functionals on A, let $N_f = \{x \in A \mid f(x^*x) = 0\}$, $N_f^{(1)} = N_f \cap Q_P$, $N_f^{(2)} = N_f \cap B_P$. Consider the inner product spaces $K_f = A/N_f$, $K_f^{(1)} = Q_P/N_f^{(1)}$, $K_f^{(2)} = B_P/N_f^{(2)}$ with the canonical inner product induced by f (e.g. on K_f, $\langle a + N_f, b + N_f \rangle = f(b^*a)$). Let $H_f, H_f^{(1)}, H_f^{(2)}$ be their respective Hilbert space completions.

Claim. $H_f = H_f^{(1)} = H_f^{(2)}$ modulo-unitary equivalence. Indeed, the natural injections $K_f^{(2)} \to K_f^{(1)} \to K_f$ define isometric embeddings of the preceding into the following. Hence they extend to linear isometries $u_1 \colon H_f^{(2)} \to H_f^{(1)}$, $u_2 \colon H_f^{(1)} \to H_f$. It suffices to prove that $u_2 \circ u_1$ is onto. For this, given $x \in A$, Theorem 1.5 (a) gives a sequence (x_u) in B_P such that $x_u \to x$. Now A, being complete and bornological, is barrelled; and hence hypocontinuous, and so the multiplication is sequentially jointly continuous. Thus $(x_u - x)^*(x_u - x) \to 0$. By [14, §8], each $f \in P(A)$ is t-continuous. Hence $x_u + N_f \to x + N_f$ in K_f. Thus $(u_2 \circ u_1)(K_f^{(2)})$ is dense in K_f. This gives $(u_2 \circ u_1)(H_f^{(2)}) = H_f$ establishing the claim.

Now $H^{(2)} \equiv \Sigma^{\oplus} H_f^{(2)} \simeq \Sigma^{\oplus} H_f^{(1)} \equiv H^{(1)} \simeq H \equiv \Sigma^{\oplus} H_f$. Let π_f (respectively $\pi_f^{(1)}$, $\pi_f^{(2)}$) be the representations of A (respectively Q_P, B_P) constructed from f on H_f (respectively $H_f^{(1)}$, $H_f^{(2)}$) by the GNS construction [24, Theorem 6.3]. These are in general unbounded representations with domains $D(\pi_f) = K_f$, $D(\pi_f^{(1)}) = K_f^{(1)}$, $D(\pi_f^{(2)}) = K_f^{(2)}$, and their closures [24, Lemma 2.6] are designated by $(\pi_f, D(\pi_f), H_f)^-$, $(\pi_f^{(1)}, D(\pi_f^{(1)}), H_f^{(1)})^-$ and $(\pi_f^{(2)}, D(\pi_f^{(2)}), H_f^{(2)})^-$. These are strongly cyclic representations respectively of A, Q_P and B_P having strongly cyclic vectors $\xi_f = 1 + N_f$, $\xi_f^{(1)} = 1 + N_f^{(1)}$, $\xi_f^{(2)} = 1 + N_f^{(2)}$ respectively. By forming their direct sum, consider the representations $(\pi, D(\pi), H) = \Sigma^{\oplus}(\pi_f, D(\pi_f), H_f)^-$ of A on H, $(\pi^{(1)}, D(\pi^{(1)}), H^{(1)}) = \Sigma^{\oplus}(\pi_f^{(1)}, D(\pi_f^{(1)}), H_f^{(1)})^-$ of Q_P on $H^{(1)}$ and $(\pi^{(2)}, D(\pi^{(2)}), H^{(2)}) = \Sigma^{\oplus}(\pi_f^{(2)}, D(\pi_f^{(2)}), H_f^{(2)})^-$ of B_P on $H^{(2)}$.

Further, Let $D(\pi_f') = \{\pi_f(Q_P)\xi_f\}$, a subspace of K_f. Let H_f' be the norm closure of $D(\pi_f')$ in H_f. Then

$$\pi_f' = \pi_f|_{D(\pi_f')}$$

defines a representation of Q_P on H_f' with domain $D(\pi_f')$. Let $(\pi_f', D(\pi_f'), H_f')$ be its closure (as representation of Q_P). It is a closed strongly cyclic representation of Q_P with strongly cyclic vector ξ_f'. Now consider the following results due to Inoue [22, Proposition 3.12].

Lemma. (a) Every closed *representation of a locally convex GB^*-algebra is self-adjoint.

(b) If $(\pi_1, D(\pi_1), H_1)$ and $(\pi_2, D(\pi_2), H_2)$ be two strongly cyclic self-adjoint representations of a *algebra A, then π_1 and π_2 are unitarily equivalent if and only if $(\pi_1(x)\xi_1, \xi_1) = (\pi_2(x)\xi_2, \xi_2)$ $(x \in A)$ where ξ_1 and ξ_2 are strongly cyclic vectors for π_1 and π_2.

It follows from this that $(\pi_f^r, D(\pi_f^r), H_f^r)$ and $(\pi_f^{(1)}, D(\pi_f^{(1)}, H_f^{(1)})^-$ are unitarily equivalent, hence so are $(\pi^{(r)}, D(\pi^{(r)}), H^{(r)}) = \Sigma^{\oplus}(\pi_f^{(r)}, D(\pi_f^{(r)}), H_f^{(r)})$ and $(\pi^{(1)}, D(\pi^{(1)}), H^{(1)})$. In what follows, we shall identify unitarily equivalent spaces.

Now let $X = \Pi H_f$, product of the Hilbert space H_f's with the product topology that is determined by the calibration $\Gamma = \{p_f | f \in P(A)\}$ where $p_f(\eta) = \|\eta_f\|_f$ for $\eta = (\eta_f) \in X$, $\|\cdot\|_f$ denoting the norm on H_f. Then the identity maps $D = D(\pi) \subsetneq H \subsetneq X$ give dense continuous embeddings. Since each $f \in P(A)$ is t-continuous, its restriction $f|_{Q_P}$ is t_p-continuous; and so by a result due to Brooks [14, Theorem 6.1], it is admissible i.e. the GNS representation $\pi_f^{(1)}$ (and so π_f^r) of Q_P determined by f is bounded mapping Q_P into bounded operators. Note that admissibility of f is equivalent to the fact that for each $x \in Q_P$, $\sup\left\{\dfrac{f(y^*x^*xy)}{f(y^*y)} \mid y \in Q_P\right\} = M_{x,f}$ (say) $< \infty$. As in [14, Theorem 7.11]. π faithfully represents, as $x \to \pi(x)$: $D(\pi) \to D(\pi)$ by $\pi(x)((\eta_f)) = (\pi_f(x)\eta_f)$, the algebra A as an EC^*-algebra on $D(\pi)$ in H such that B_P is represented as $x \to \phi(x) = \overline{\pi(x)}$, as a C^*-algebra of bounded operators on H with $\phi(B_P) = \mathscr{B}(H) \cap \overline{\pi(A)}$.

Let $a \in Q_P$. We show that for each $f \in P(A)$, $p_f(\pi(a)x) \leqslant M_{a,f} p_f(x)$ for each $x \in X$ of the form $x = \{(b_g + N_g^{(1)}) | g \in P(A)\}$ with $b_g \in Q_P$ for each $g \in P(A)$. Indeed, $\pi(a)x = (ab_g + N_g^{(1)}) \in X$ and

$$\begin{aligned} p_f(\pi(a)x) &= \|ab_f + N_f^{(1)}\| \\ &= f(b_f^* a^* a b_f)^{1/2} \\ &= M_{a,f}^{1/2} f(b_f^* b_f)^{1/2} = M_{a,f}^{1/2} p_f(x). \end{aligned}$$

Now let $x \in X$ be of the form $x = (b_g + N_g)$ with $b_g \in A$ for all $g \in P(A)$. Then for each such g, by Theorem 1.5(a), $(b_g)_n = b_g(1 + (1/n)b_g^* b_g)^{-1} \in B_P \subset Q_P$ and $(b_g)_n \to b_g$. By the sequential joint continuity of multiplication and the continuity of f, we get $p_f(\pi(a)x) \leqslant M_{a,f}^{1/2} p_f(x)$. Thus $\pi(a)$ can be regarded as a Γ-quotient-bounded continuous linear operator defined on a dense subspace of X, and hence it admits a unique Γ-quotient-bounded continuous linear extension $\tilde{\pi}(a)$ on X. Thus $\tilde{\pi}(Q_P) \subset Q(X, \Gamma)$.

Further let $a \in B_P, f \in P(A)$. Consider an $x \in X$ of the form $x = \{(b_g + N_g^{(2)}) | g \in P(A)\}$ with $b_g \in B_P = A(B_0)$ for each $g \in P(A)$. Then

$$\begin{aligned} p_f(\pi(a)x) &= f(b_f^* a^* a b_f) \\ &\leqslant r_{B_P}(a^*a)^{1/2} f(b_f^* b_f) \text{ by [10, Lemma 37.6]} \\ &\leqslant \|a\|_{B_0} p_f(x) \end{aligned}$$

and so $p_f(\pi(a)x) \leqslant \|a\|_{B_0} p_f(x)$ $(x \in X,\ f \in P(A))$. Thus $\pi(a) \in B(X,\Gamma)$ and $p_\Gamma(\pi(a)) \leqslant \|a\|_{B_0}$. Also π is faithful and the spectral radius of an element in a C^*-algebra $\mathscr{A}$, whether calculated in $\mathscr{A}$ or in any Banach algebra containing $\mathscr{A}$ is the same. Hence

$$\begin{aligned}\|a\|_{B_0}^2 &= \|a^*a\|_{B_0} = r_{A(B_0)}(a^*a)\\ &= r_{B(X,\Gamma)}(\tilde{\pi}(a)^*\tilde{\pi}(a)) \leqslant p_\Gamma(\tilde{\pi}(a))^2.\end{aligned}$$

Thus $\|a\|_{B_0} (= p(a)) = p_\Gamma(\tilde{\pi}(a))$ and π isometrically maps B_P into $(B(X,\Gamma), p_\Gamma)$.

On the other hand, consider the natural LMC topology t_Γ on $Q(X,\Gamma)$ that is determined by the seminorms $T \to q_f(T) = \sup\{p_f(Tz) \mid z \in X, p_f(z) \leqslant 1\}$. To prove that π maps (Q_P, t_P) continuously into $(Q(X,\Gamma), t_\Gamma)$, it is sufficient to show that given $a \in Q_P$, $f \in P(A)$, there exists a finite set $\alpha_1, \ldots, \alpha_n$ in Δ such that $q_f(\pi(a)) \leqslant \max\limits_{1 \leqslant i \leqslant n} q_{\alpha_i}(a)$ for each $a \in Q_P$. (The argument that we give for this is based on the idea in the proof of [11, Theorem 6.1]). Since $f|_{Q_P}$ is t_P-continuous, it is bounded on some neighbourhood U of o in (Q_P, t_P). This U can be assumed to be of the form $U = \bigcap\limits_{i=1}^{n} U_{\alpha_i}$ for some $\alpha_1, \ldots, \alpha_n$ in Δ where $U_{\alpha_i} = \{y \in Q_P \mid q_{\alpha_i}(y) \leqslant 1\}$ for $i = 1, 2, \ldots, n$. Let $q = \max\limits_{1 \leqslant i \leqslant n} q_{\alpha_i}$. Since $\{q_\alpha \mid \alpha \in \Delta\}$ is a b^*-calibration for (Q_P, t_P) [9, Theorem 3.1], $\ker q = \{x \in Q_P \mid q(x) = 0\}$ is a *ideal of Q_P and $Q_P/\ker q$ is a *algebra which is a normed algebra under the norm $\tilde{q}(y + \ker q) = q(y)$. Then f_q defined as $f_q(y + \ker q) = f(y)$ is a continuous positive linear functional on $(Q_P/\ker q, \tilde{q})$. Let h on $Q_P/\ker q$ be defined as $h(z + \ker q) = f_q((b + \ker q)^*(z + \ker q)(b + \ker q))$ for a fixed b in Q_P. Then h is a continuous positive linear functional satisfying $|h(z + \ker q)| \leqslant h(1 + \ker q)\tilde{q}(z + \ker q)$. Hence

$$\begin{aligned}f(b^*z^*zb) &= f_q((b + \ker q)^*(z + \ker q)^*(z + \ker q)(b + \ker q))\\ &= h((z + \ker q)^*(z + \ker q))\\ &= h(1 + \ker q)\tilde{q}((z + \ker q)^*(z + \ker q))\\ &= f(b^*b)(\tilde{q}(z + \ker q))^2\\ &= f(b^*b)(q(z))^2\end{aligned}$$

for all $z \in Q_P$. Hence

$$\begin{aligned}M_{z,f}^{1/2} &= \sup\left\{\frac{f(b^*z^*zb)^{1/2}}{f(b^*b)^{1/2}} \mid b \in Q_P\right\}\\ &\leqslant q(z).\end{aligned}$$

Hence $q_f(\tilde{\pi}(z)) \leqslant M_{z,f}^{1/2} \leqslant \max\limits_{1 \leqslant i \leqslant n} q_{\alpha_i}(z)$ for all $z \in Q_P$, which means that π maps (Q_P, t_P) continuously into $(Q(X,\Gamma), t_\Gamma)$.

Note also that $\pi(Q_P)$ is *isomorphic to $\pi^r(Q_P)$. As π^r is a direct sum of bounded representations, it follows that $\pi^r(Q_P)$ is a weakly unbounded operator algebra. This completes the proof.

3. Examples

In this section, we analyse a few examples with a view of illustrating quotient-bounded elements and proper GB^*-algebras. First two of the following examples have been analyzed in detail in [6] and [7]. Since some of our examples are motivated by them, we have briefly included the relevant constructions.

3.1. *Arens' algebra $L^\omega(X)$ on a finite measure space*

Let (X, Σ, μ) be a finite measure space. Let $L^\omega(X) = \bigcap_{1 \leqslant p < \infty} L^p(X)$. It is a *algebra with pointwise operations. Let τ^ω be the locally convex topology on $L^\omega(X)$ defined by the calibration $P = \{\|\cdot\|_p \mid 1 \leqslant p < \infty\}$. Then $(L^\omega(X), \tau^\omega)$ is a complete metrizable locally convex GB^*-algebra the underlying C^*-algebra being $L^\infty(X)$.

Assertions: [9, Example 4.1]

(1) $Q_P = B_P = L^\infty(X)$

(2) $L^\omega(X)$ is not a proper GB^*-algebra. In fact, there is no b^*-algebra containing $L^\infty(X)$ and properly contained in $L^\omega(X)$.

3.2. *Arens' algebra L^ω_{loc} on a σ-finite measure space*

On an infinite σ-finite, nonatomic measure space (X, Σ, μ) with $X = \bigcup_1^\infty X_n$, $X_n \in \Sigma$, $X_n \subset X_{n+1}$, $\mu(X_n) < \infty$ for all n; let $L^\omega_{\text{loc}}(X) = \bigcap_{1 \leqslant p < \infty} L^p_{\text{loc}}(X)$ with the locally convex topology defined by

$$P = \{\|\cdot\|_{k,p} \mid k, p \in \mathbf{N}\} \text{ where } \|f\|_{k,p} = \left(\int_{X_k} |f|^p \, d\mu\right)^{1/p}.$$

Then $L^\omega_{\text{loc}}(X)$ is a locally convex GB^*-algebra with unit ball $B_0 = \{f \in L^\infty(X) \mid \|f\|_\infty \leqslant 1\}$.

Assertions: [9, Example 4.3]

(1) $B_P = L^\infty(X)$ and $Q_P = L^\infty_{\text{loc}}(X)$, the topology t_P being determined by $\|f\|_{\infty,k} = \text{ess. sup.}_{x \in X_k} |f(x)|$.

(2) $L^\omega_{\text{loc}}(X)$ is a proper GB^*-algebra.

3.3. *Hardy–Arens algebra $H^\omega(U)$*

Let $U = \{z \in \mathbf{C} \mid |z| < 1\}$. Let $H^p(U)$ $(1 \leqslant p < \infty)$ be the usual Hardy space of function on U. Let $H^\omega(U) = \bigcap_{1 \leqslant p < \infty} H^p(U)$. It is a locally convex *algebra with pointwise operations (the involution being $f^*(z) = \overline{f(\bar{z})}$) and the topology determined by the calibration $P = \{\|\cdot\|_p \mid 1 \leqslant p < \infty\}$. It properly contains the Banach algebra $H^\infty(U)$.

The algebra H^ω is not symmetric, hence not GB^*; however $Q_P = B_P = H^\infty(U)$.

Let $\Gamma = \{z \in \mathbb{C} \mid |z| = 1\}$. As usual, $H^p(U)$ can be identified with a subalgebra of the convolution algebra $L^p(\Gamma)$. Then $H^\omega(U)$ is a closed subalgebra of the convolution algebra $L^\omega(\Gamma)$. In this case $(L^\omega(\Gamma), \tau^\omega)$ is a complete LMC algebra, and $Q_P = L^\omega(\Gamma)$, $B_P = L^\infty(\Gamma)$. Similarly for $H^\omega(U)$, $Q_P = H^\omega(U)$, $B_P = H^\infty(U)$.

For $1 \leqslant p < \infty$, we can also consider the Banach algebra $AC_p[0, 1] = \{f \in C[0, 1] \mid f'$ exists a.e. and $f' \in L^p[0, 1]\}$ with norm $\|f\|_{(p)} = \|f\|_\infty + \|f'\|_p$; and construct the algebra $AC_\omega[0, 1]$.

Now we shall consider non-commutative analogues of these, which we analyze in more details.

3.4. *Non-commutative Arens' algebra $L^\omega(\mathscr{A}, \varphi)$ on a regular probability gauge space*

Let Γ be a regular probability gauge space [31] viz a triple $\Gamma = (H, \mathscr{A}, \varphi)$ where H is a complex Hilbert space, $\mathscr{A}$ a von Neumann algebra acting on H and φ a faithful normal tracial state on $\mathscr{A}$. For $A \in \mathscr{A}$, let $\|A\|_p = (\varphi(|A|^p))^{1/p} (1 \leqslant p < \infty)$ where $|A| = (A^*A)^{1/2}$. It is norm on $\mathscr{A}$. Let $L^p(\mathscr{A}, \varphi)$ be the completion of $(\mathscr{A}, \|\cdot\|_p)$. For $p = \infty$, let $L^\infty(\mathscr{A}, \varphi) = \mathscr{A}$ with the operator norm. These spaces can be identified as subsets of the set of closed operators in H affiliated with $\mathscr{A}$. For $1 \leqslant r \leqslant p$, $L^\infty(\mathscr{A}, \varphi) \subset L^p(\mathscr{A}, \varphi) \subset L^r(\mathscr{A}, \varphi) \subset L^1(\mathscr{A}, \varphi)$. By using non-commutative Holder's inequality, it can be shown that $L^\omega(\mathscr{A}, \varphi) = \bigcap_{1 \leqslant p < \infty} L^p(\mathscr{A}, \varphi)$ is a *algebra with identity with strong operations $(S+T)^-$, $(\lambda T)^-$, $(ST)^-$ and the operator adjoint as the involution. The following summarizes some aspects of the structure of $L^\omega(\mathscr{A}, \varphi)$. Let τ^ω be the topology on L^ω defined by $P = \{\|\cdot\|_p \mid 1 \leqslant p < \infty\}$.

3.4a *Proposition*: (a) $(L^\omega(\mathscr{A}, \varphi), \tau^\omega)$ is a complete metrizable locally convex GB^*-algebra with unit ball $B_0 = \{A \in \mathscr{A} \mid \|A\| \leqslant 1\}$.

(b) The positive cone in $L^\omega(\mathscr{A}, \varphi)$ is normal and $\tau^\omega = T$, the largest locally convex GB^*-topology on $L^\omega(\mathscr{A}, \varphi)$ [14, § 6]

$= \pi(T)$ the largest locally convex GB^*-topology with normal positive cone [8].

(c) A subset $\mathscr{L}$ of $L^\omega(\mathscr{A}, \varphi)$ is a closed left ideal of $L^\omega(\mathscr{A}, \varphi)$ if and only if $\mathscr{L} = L^\omega(\mathscr{A}, \varphi)K$ where K is a projection in $\mathscr{A}$. Further it is a closed maximal left ideal if and only if $1 - K$ is a minimal projection in $\mathscr{A}$.

(d) Every topologically primitive ideal $\mathscr{P}$ (i.e. quotient of a closed maximal left ideal) is of the form $\mathscr{P} = L^\omega(\mathscr{A}, \varphi)(1 - C(K))$ where $C(K)$ is the central support of K in $\mathscr{A}$.

Proof. (a) The topology τ^ω is also determined by the calibration $P = \{\|\cdot\|_n \mid n = 1, 2, \ldots\}$, hence is metrizable. It is also complete. Theorem 1.5(b) implies that $L^\omega(\mathscr{A}, \varphi)$ is a GB^*-algebra with the underlying C^*-algebra $\mathscr{A}$.

(b) For A, B in $L^p(\mathscr{A}, \varphi)$, $0 \leqslant A \leqslant B$ gives $\|A\|_p \leqslant \|B\|_p$ in view of [33, Proposition 2.5 (iv)]. Hence by [29, Ch. V, § 3.1], the positive cone in $L^\omega(\mathscr{A}, \varphi)$ is normal. Thus every continuous hermitian linear functional in L^ω is a difference of two positive functionals. This with the automatic continuity of a positive functional on a complete metrizable *algebra with 1 [13, p. 178] implies that the dual of $L^\omega(\mathscr{A}, \varphi)$ is $(L^\omega(\mathscr{A}, \varphi))^P$, the

complex linear span of all positive functionals. Since $L^\omega(\mathscr{A}, \varphi)$ is Mackey, it follows from [14, §8] that $\tau^\omega = T$, and hence $\tau^\omega = \pi(T)$.

In fact, the dual of $L^\omega(\mathscr{A}, \varphi)$ is identified with $\bigcup_{1 < q \leqslant \infty} L^q(\mathscr{A}, \varphi)$, and $L^\omega(\mathscr{A}, \varphi)$ is reflexive.

(c) and (d) can be verified either by a modification of the standard W^*-algebra techniques [28, Proposition 1.10.1] or else by applying [5, Corollary 3].

Assertions: (a) Taking $P = \{\|\cdot\|_n \mid n = 1, 2, \ldots\}$, $Q_P = B_P = \mathscr{A}$

(b) $L^\omega(\mathscr{A}, \varphi)$ is not a proper GB^*-algebra.

Proof. (a) It suffices to show that $Q_P \subset \mathscr{A}$. For, then $Q_P = B_P = \mathscr{A}$, since $\mathscr{A} \subset B_P$; and further as P is countable, the b^*-topology on Q_P is also metrizable, and the open mapping theorem shows that the topologies on these algebras are identical.

Let $A \in Q_P$. Then for some scalar $q_1(A)$, $\varphi(|AX|) \leqslant q_1(A)\varphi(|X|)$ for all $X \in L^\omega(\mathscr{A}, \varphi)$. As $L^\omega(\mathscr{A}, \varphi)$ is dense in $L^1(\mathscr{A}, \varphi)$, this holds for all $X \in L^1(\mathscr{A}, \varphi)$. Also, for each $Y \in L^1(\mathscr{A}, \varphi)$,

$$\varphi(|Y|) = \|Y\|_1 = \sup\{|\varphi(YS)| \mid S \in L^\infty(\mathscr{A}, \varphi), YS \in L^1(\mathscr{A}, \varphi) \|S\|_\infty \leqslant 1\}$$
$$\geqslant |\varphi(Y)|.$$

Hence $|\varphi(AX)| \leqslant q_1(A)\|X\|_1$ $(X \in L^1(\mathscr{A}, \varphi))$. Hence by the noncommutative Radon–Nikodym Theorem, $A \in L^\infty(\mathscr{A}, \varphi) = \mathscr{A}$.

(b) Suppose that there exists a *subalgebra Q of $L^\omega(\mathscr{A}, \varphi)$ admitting a b^*-topology t and containing $L^\infty(\mathscr{A}, \varphi)$ such that $(L^\infty(\mathscr{A}, \varphi), \|\cdot\|_\infty) \to (Q, t) \to (L^\omega(\mathscr{A}, \varphi), \tau^\omega)$ are continuous injections. Now φ being a continuous positive linear form on $L^\omega(\mathscr{A}, \varphi)$, $\psi = \varphi|_Q$ is t-continuous. Let (π_ψ, H_ψ) be the cyclic GNS representation of Q on a Hilbert space H_ψ associated with ψ. By [11, Theorem 6.1], ψ is admissible i.e.

$$\sup\left\{\frac{\psi(Y^*X^*XY)}{\psi(Y^*Y)} \mid Y \neq 0, Y \in Q\right\} < \infty;$$

π_ψ maps elements of Q_P as bounded operators on H_ψ with

$$\|\pi_\psi(X)\|^2 = \sup\left\{\frac{\psi(Y^*X^*XY)}{\psi(Y^*Y)} \mid Y \neq 0, Y \in Q\right\}$$

and $\pi_\psi\colon (Q, t) \to (\mathscr{B}(H_\psi), \|\cdot\|)$ is continuous. Let 1_ψ be the unit cyclic vector in H_ψ for π_ψ. Then $\psi(X) = (\pi_\psi(X)1_\psi, 1_\psi)$ $(X \in Q)$. Since φ is faithful, 1_ψ is separating for π_ψ; hence π_ψ is faithful. This forces $Q = \mathscr{A}$.

3.5 *Noncommutative Hardy–Arens algebra*

Let $\mathscr{A}$ be a finite von Neumann algebra acting on a Hilbert space H. Let $\{\alpha_t \mid t \in \mathbf{R}\}$ be a flow on $\mathscr{A}$ i.e. a σ-weakly continuous one-parameter group of *automorphisms on $\mathscr{A}$. Let φ be a faithful, α_t-invariant, normal tracial state on $\mathscr{A}$. By [27, Proposition 2.2], for each p, $1 \leqslant p < \infty$, $\{\alpha_t \mid t \in \mathbf{R}\}$ extends uniquely to a strongly continuous representation of $\mathbf{R}$ on isometries of $L^p(\mathscr{A}, \varphi)$, to be denoted by $\{\alpha_t \mid t \in \mathbf{R}\}$. This defines a

representation of $L^1(\mathbf{R})$ on the Banach space $L^p(\mathscr{A}, \varphi)$ as $\alpha(f)T = \int_{-\infty}^{\infty} f(t)\alpha_t(T)\,dt$ ($T \in L^p(\mathscr{A}, \varphi)$, $f \in L^1(\mathbf{R})$). For $f \in L^1(\mathbf{R})$, let $Z(f) = \{t \in \mathbf{R} \mid \hat{f}(t) = 0\}$, $\hat{f}$ being the Fourier transform of f. Let $Sp_\alpha(T) = \cap\{Z(f) \mid f \in L^1(\mathbf{R}), \alpha(f)T = 0\}$. The associated noncommutative Hardy spaces are defined as: For $1 \leqslant p \leqslant \infty$, $H^p(\alpha) = \{T \in L^p(\mathscr{A}, \varphi) \mid Sp_\alpha(T) \subset [0, \infty)\}$. Then $H^\infty(\alpha)$ is a nonself-adjoint closed subalgebra of $\mathscr{A}$.

Let $H^\omega(\alpha) = \bigcap_{1 \leqslant p < \infty} H^p(\alpha)$, a complete locally convex algebra with relative topology induced by τ^ω on $L^\omega(\mathscr{A}, \varphi)$. Taking $P = \{\|\cdot\|_p \mid p \in \mathbf{N}\}$, we have $Q_P = B_P = H^\infty(\alpha)$.

3.6 *Maximal-unbounded Hilbert algebras* [21]

Let $\mathscr{D}$ be an inner product space that is a *algebra. Assume the following:

(a) $\langle \xi, \eta \rangle = \langle \eta^*, \xi^* \rangle$,

(b) $\langle \xi\eta, \nu \rangle = \langle \eta, \xi^*\nu \rangle$,

(c) Let π and π' on $\mathscr{D}$ be defined as $\pi(\xi)\eta = \pi'(\eta)\xi = \xi\eta$. Let H be the completion of $\mathscr{D}$. Let $\mathscr{D}_0 = \{\xi \in \mathscr{D} \mid \pi(\xi) \text{ is bounded}\}$. Then $\mathscr{D}_0^2$ is assumed to be dense in $\mathscr{D}$. Such a $\mathscr{D}$ is called an *unbounded Hilbert algebra.*

For each $x \in H$, let $\pi_0(x)\xi = \overline{\pi_0'(\xi)}x$, $\pi_0'(x)\xi = \overline{\pi_0(\xi)}x$ ($\xi \in \mathscr{D}_0$). Here on $\mathscr{D}_0$, $\pi_0 = \pi|_{\mathscr{D}_0}$, $\pi_0' = \pi'|_{\mathscr{D}_0}$. Let $(\mathscr{D}_0)_b = \{x \in H \mid \pi_0(x) \in \mathscr{B}(H)\}$, a Hilbert algebra. Let $\mathscr{u}_0(\mathscr{D}_0)$ be the left von Neumann algebra of $\mathscr{D}_0$ and let φ be the natural semifinite trace on $\mathscr{u}_0(\mathscr{D}_0)^+$. Let $\mathscr{m}(\varphi)$ (respectively $\mathscr{m}(\mathscr{u})$) denote the extended W^*-algebra of operators in H that are φ-restrictedly measurable (respectively measurable with respect to $\mathscr{u}_0(\mathscr{D}_0)$). For $T \in \mathscr{m}(\varphi)$, let $|T| = (T^*T)^{1/2}$, $\mu(T) = \sup\{\varphi(\overline{\pi_0(\xi)}) \mid 0 \leqslant \pi_0(\xi) \leqslant T, \xi \in (\mathscr{D}_0)_b^2\}$. For $1 \leqslant p < \infty$, let $L^p(\varphi) = \{T \in \mathscr{m}(\mathscr{u}_0) \mid \|T\|_p = \mu(|T|^p)^{1/p} < \infty\}$. For $p \geqslant 2$, $L_2^p(\mathscr{D}_0) = \{x \in H \mid \overline{\pi_0(x)} \in L^p(\varphi)\}$ with $\|x\|_p = \|\pi_0(x)\|_p$. Let $L_2^\omega(\varphi) = \bigcap_{2 \leqslant p < \infty} L^p(\varphi)$, $L_2^\omega(\mathscr{D}_0) = \{x \in H \mid \overline{\pi_0(x)} \in L_2^\omega(\varphi)\} = \bigcap_{2 \leqslant p < \infty} L_2^p(\mathscr{D}_0) = \bigcap_{n=2}^{\infty} L_2^n(\mathscr{D}_0)$. Let $\|x\|_\infty = \|\pi_0(x)\|_\infty$ for x in $L_2^\infty(\mathscr{D}_0) = (\mathscr{D}_0)_b$.

The topology τ_2^ω is defined on $L_2^\omega(\mathscr{D}_0)$ by $\{\|\cdot\|_p \mid 2 \leqslant P < \infty\}$, or equivalently by $P = \{\|\cdot\|_n \mid n = 2, 3, \ldots\}$–making $L_2^\omega(\mathscr{D}_0)$ a complete metrizable locally convex *algebra which is a GB^*-algebra with $((\mathscr{D}_0)_b, \|\cdot\|_\infty)$ as the underlying C^*-algebra. The algebra $L_2^\omega(\mathscr{D}_0)$ is a maximal unbounded Hilbert algebra containing $\mathscr{D}_0$. (We assume that $\mathscr{D}_0$ contains identity 1.)

Assertions: (a) For $L_2^\omega(\mathscr{D}_0)$ and P as above, $Q_P = B_P = (\mathscr{D}_0)_b$.

(b) For $L_2^\omega(\varphi)$ with analogous P, $Q_P = B_P = \mathscr{u}_0(\mathscr{D}_0)$.

Proof. We supply details of (a). As in example (3.4), it is sufficient to show that $Q_P \subset B_P = (\mathscr{D}_0)_b$. Let $a \in (\mathscr{D}_0)_b$. Then $\overline{\pi_0(a)} \in \mathscr{B}(H)$; and for each $x \in L_2^\omega(\mathscr{D}_0)$, $\|ax\|_p = \|\overline{\pi_0(ax)}\|_p \leqslant \|\overline{\pi_0(a)}\| \, \|\overline{\pi(x)}\|_p = \|a\|_\infty \|x\|_p$ giving $a \in B_p$, $(\mathscr{D}_0)_b \subset B_p$, $p(a) \leqslant \|a\|_\infty$.

Let $a \in Q_P$. Then for each p, $2 \leqslant p < \infty$, $\|ax\|_p \leqslant a_p(a)\|x\|_p$ $(x \in L_2^\omega(\mathscr{D}_0))$. Taking $p = 2$ and using the fact that $L_2^2(\mathscr{D}_0)$ is identified with H, it follows that $\overline{\pi_0(a)}$ is a norm bounded operator on $L_2^\omega(\mathscr{D}_0)$. Let π_2^ω be the left regular representation of $L_2^\omega(\mathscr{D}_0)$. Then $\pi_2^\omega(a)$ is a bounded operator. But by [21 (II), p. 422], the algebra $\mathscr{A}(L_2^\omega(\mathscr{D}_0))$ generated by $\pi_2^\omega(L_2^\omega(\mathscr{D}_0))$ and $\mathscr{A}_0(\mathscr{D}_0)|_{L_2^\omega(\mathscr{D}_0)}$ is an EW^*-algebra over $\mathscr{A}_0(\mathscr{D}_0)$. Hence $\overline{\pi(a)} \in \mathscr{A}_0(\mathscr{D}_0)$. Hence $a \in (\mathscr{D}_0)_b$. Thus $Q_P \subset (\mathscr{D}_0)_b$, $Q_P = B_P = (\mathscr{D}_0)_b$. The topologies also agree.

Remark. It would be interesting to investigate a noncommutative analogoue of (3.2).

3.7 *L^ω- and L^ω_{loc}-integrals of a field of C^*-algebras*

Below we briefly describe a construction of auxilliary nature which may be of some independent interest.

(A) Let (X, Σ, μ) be a nonatomic finite measure space. Let $t \to H(t)$ be a measurable field of separable Hilbert spaces with $H = \int_X^\oplus H(t)\mathrm{d}\mu(t)$. Let $t \to \mathscr{A}(t)$ be a measurable field of C^*-algebra with $\mathscr{A}(t)$ acting on $H(t)$. Let $\mathscr{A} = \int^\oplus \mathscr{A}(t)\,\mathrm{d}\mu(t)$ realized as a C^*-algebra on H. For each p, $1 \leqslant p < \infty$, define the 'L^p-integral' of $t \to \mathscr{A}(t)$ as

$$L^p - \int_X^\oplus \mathscr{A}(t)\,\mathrm{d}\mu(t) = \text{collection of all measurable operator } T: t \to T(t) \in \mathscr{A}(t) \text{ such that}$$

$$\|T\|_p = \left(\int_X \|T(t)\|^p \,\mathrm{d}\mu(t)\right)^{1/p} < \infty.$$

It is a Banach space. For $1 \leqslant r < p < \infty$,

$$L^r - \int_X^\oplus \mathscr{A}(t)\,\mathrm{d}\mu(t) \supset L^p - \int_X^\oplus \mathscr{A}(t)\,\mathrm{d}\mu(t) \supset \mathscr{A} = L^\infty - \int_X^\oplus \mathscr{A}(t)\,\mathrm{d}\mu(t).$$

Let

$$L^\omega - \int_X^\oplus \mathscr{A}(t)\,\mathrm{d}\mu(t) = \bigcap_{1 \leqslant p < \infty} L^p - \int_X^\oplus \mathscr{A}(t)\,\mathrm{d}\mu(t).$$

With usual operations on operator fields, it is a *algebra with 1 which is a complete metrizable locally convex GB^*-algebra under the topology τ^ω defined by $P = \{\|\cdot\|_p \mid 1 \leqslant p < \infty\}$, the unit ball being $B_0 = \{T: t \to T(t) \text{ in } \mathscr{A} \mid \|T\| = \operatorname{ess\,sup}\|T(t)\| \leqslant 1\}$. We conjecture that $Q_P = B_P = \mathscr{A}$.

(B) Let X be a σ-finite measure space as in (3.2). Define

$$L^p_{\text{loc}} - \int_X^\oplus \mathscr{A}(t)\,\mathrm{d}\mu(t) = \text{collection of all measurable operator fields } T: t \to T(t) \in \mathscr{A}(t)$$

such that for each $n = 1, 2, 3, \ldots$

$$\|T\|_{p,n} = \left(\int_{X_n} \|T(t)\|^p \,\mathrm{d}\mu(t)\right)^{1/p} < \infty.$$

It is a Frechet space with the topology τ^p_{loc} defined by $P_p = \{\|\cdot\|_{p,n} | n = 1, 2, \ldots\}$. Note that

$$L^\infty_{\text{loc}} - \int_X^{\oplus} \mathscr{A}(t)\,d\mu(t) = \text{collection of all measurable operator fields } T\colon t \to T(t) \in \mathscr{A}(t)$$
$$\text{such that for } n = 1, 2, \ldots \|T\|_{\infty,n} = \operatorname*{ess\,sup}_{X_n} \|T(t)\| < \infty.$$

It is a b^*-algebra with a b^*-calibration $P_\infty = \{\|\cdot\|_{\infty,n} | n = 1, 2, \ldots\}$. Now we can formulate, as in (3.2),

$$L^\omega_{\text{loc}} - \int_X^{\oplus} \mathscr{A}(t)\,d\mu(t) = \bigcap_{1 \leqslant p < \infty} L^p_{\text{loc}} - \int_X^{\oplus} \mathscr{A}(t)\,d\mu(t) = A \text{ (say)}.$$

We conjecture that A is a GB^*-algebra, P being a GB^*-calibration and $B_P = \mathscr{A}$, $Q_P = L^\infty_{\text{loc}} - \int_X^{\oplus} \mathscr{A}(t)\,d\mu(t)$.

Remark. To the best of our knowledge, the above L^p-integrals of operator fields do not seem to have been investigated in the literature. We do not know whether their elements can be regarded as concrete operators in H.

3.8 *A convolution algebra of periodic distributions*

Let $T = \{z \in \mathbb{C} | \, |z| = 1\}$ be the unit circle. Let $D(T)$ be the cummutative convolution *algebra with identity of all distributions on T, the identity being the Dirac delta δ and the involution being $u \to u^*$ defined as $\langle u^*, \phi \rangle = \overline{\langle u, \phi^* \rangle}$, $\phi^*(z) = \overline{\phi(\bar{z})}$ ($\phi \in C^\infty(T)$, $z \in T$). Let $u \to \hat{u}$, $\hat{u}(n) = \langle u, \exp(-\text{int}) \rangle$ ($n \in \mathscr{Z}$, the integers) be the Fourier–Schwartz transform that maps $D(T)$ *isomorphically onto the *algebra $A = \{(a_n)_{-\infty}^{\infty} | a_n \in \mathbb{C}$, $a_n = 0(|n|^m)$ for some m, depending in general on $(a_n)\}$ with pointwise operations and the complex conjugation as the involution. Under this map, the *subalgebra $\mathbf{PM}(T)$ (pseudomeasures on T) of $D(T)$ is mapped onto $1^\infty(\mathscr{Z})$, the algebra of bounded sequences on $\mathscr{Z}$. The algebra $\mathbf{PM}(T)$ with norm $\|u\| = \|\hat{u}\|_\infty$ is a W^*-algebra.

3.8a. *Proposition.*

The algebra $D(T)$ with the weak topology $\sigma = \sigma(D(T), C^\infty(T))$ is a sequentially complete locally convex GB^*-algebra with unit ball $B_0 = \{u \in \mathbf{PM}(T) | \, \|u\| \leqslant 1\}$.

Proof. Using [15, § 12.6], it is a simple matter to verify that $(D(T), \sigma)$ is a locally convex algebra with continuous involution and jointly continuous multiplication.

Claim. $\mathbf{PM}(T) = (D(T), \sigma)_0$, the set of (Allan) bounded elements of $(D(T), \sigma)$ [1, § 2].

Let $A(T)$ be the Wiener algebra viz the algebra of continuous functions on T having absolutely convergent Fourier series. Given $u \in \mathbf{PM}(T)$, $\|u\| \leqslant \lambda$; the set $S = \{(\lambda^{-1}u)^n | n = 1, 2, \ldots\}$ is σ ($\mathbf{PM}(T)$, $A(T)$) bounded and so is σ-bounded. (Note that $\mathbf{PM}(T)$ is the dual of $A(T)$). Conversely, if $u \in D(T)$ is a bounded element of $(D(T), \sigma)$, then there is

a $\lambda \neq 0$ in $\mathbb{C}$ such that for each $\phi \in C^\infty(T)$, there is a constant M_ϕ such that $|\langle (\lambda^{-1}u)^n, \phi \rangle| \leqslant M_\phi$ for all $n = 1, 2, \ldots$ Taking $\exp(-ikt)$ (for some $k \in \mathscr{Z}$) as ϕ, it follows that $|[\hat{u}(k)]/\lambda| \leqslant M_k^{1/n}$ for all n. Thus $|\hat{u}(k)| \leqslant \lambda$ for all $k \in \mathscr{Z}$, $\hat{u} \in l^\infty(\mathscr{Z})$ and so $u \in \mathrm{PM}(T)$. Thus $\mathrm{PM}(T) = (D(T), \sigma)_0$.

Now it is easily seen that $D(T)$ is symmetric and $B_0 = \{u \in \mathrm{PM}(T) \mid \|u\| \leqslant 1\}$ is the greatest member of $\mathscr{B}^*$. The sequential completeness of σ follows from [26, Theorem 2.8]. This completes the proof.

Note that $(D(T), \sigma)$ is not complete in view of [30, Theorem 32.2]. The algebra $D(T)$ admits two other important topologies.

(a) The topology σ' defined by the seminorms $p_n(u) = |\hat{u}(n)|$ for $n \in \mathscr{Z}$. Then $(D(T), \sigma')$ is a metrizable LMC*-algebra which is also a GB^*-algebra with unit ball B_0.

(b) The strong topology $\beta = \beta(D(T), C^\infty(T))$ of uniform convergence on all bounded subsets of the Frechet space $C^\infty(T)$.

The importance of β is revealed by the following. Note that the positive elements of $D(T)$ are

$$D(T)^+ = \{u \in D(T) \mid \langle u, \phi * \phi^* \rangle \geqslant 0 \text{ for all } \phi \in C^\infty(T)\}.$$

3.8b *Proposition*

(i) The algebra $(D(T), \beta)$ is a complete locally convex GB^*-algebra with unit ball $B_0 = \{u \in \mathrm{PM}(T) \mid \|u\| \leqslant 1\}$.

(ii) Every positive linear functional f on $D(T)$ is β-continuous, and is of the form $f(u) = \langle u, \phi^* * \phi \rangle$ for some $\phi \in C^\infty(T)$.

(iii) The strong topology β

$= T$ the largest locally convex GB^*-topology

$= \tau\langle D(T), D(T)^P \rangle$, the Mackey topology of the duality $\langle D(T), D(T)^P \rangle$, $D(T)^P$ being the linear span of positive functionals on $D(T)$

and

$\sigma = \sigma(D(T), D(T)^P)$, the positive functional topology as discussed in [14, § 8] and [2, § 5].

(iv) The positive cone in $D(T)$ is β-normal, and $\beta = \pi(T)$, the topology that $D(T)$ admits via its universal representation as an OP^*-algebra with uniform topology [8].

Proof. Since $C^\infty(T)$ is Frechet (with its usual topology), the completeness of $(D(T), \beta)$ follows from [30, Corollary 2 to Theorem 32.2]. The rest of (i) is a routine. Further, $(D(T), \beta)$ being the strong dual of a reflexive Frechet space, is bornological [29, Ch. IV, § 6.6 Corollary 1], hence barrelled [29, Ch. IV § 6.6]. By [14, Corollary 8.2], every positive functional on $D(T)$ is β-continuous. This gives (ii). Since $(D(T), \beta)' = (D(T))^P \simeq C^\infty(T)$ due to reflexivity of $C^\infty(T)$, $\beta = \tau(D(T), D(T)^P) =$ Dixon topology, the last equality being a consequence of the fact that in a commutative GB^*-algebra A, Dixon topology $= \tau(A, A^P)$. Finally $D(T)$ being a vector lattice, and $(D(T), \beta)' = D(T)^P$, the positive cone is β-normal by [29, Ch. V § 6.4]. Since $\pi(T)$ is the largest locally convex GB^*-topology with normal positive cone, $\beta = \pi(T)$. This completes the proof.

Since $(\mathrm{PM}(T), \|\cdot\|)$ is a W^*-algebra with predual $A(T)$ and the identity map id:

$(\mathrm{PM}(T), \sigma(\mathrm{PM}(T), A(T))) \to (D(T), \sigma)$ is continuous, the fact that β and σ have the same dual, and hence they determine same sets of closed ideals, gives the following by applying [5, Corollary 3].

3.8c *Proposition*

(a) A subset I of $D(T)$ is a β-closed ideal of $D(T)$ if and only if $I = I_K$ for some non-empty subset K of $\mathscr{Z}$ where

$$I_K = \{u \in D(T) \mid \hat{u}(n) = 0 \text{ for all } n \in K\}$$

(b) A linear functional f on $D(T)$ is multiplicative if and only if $f = f_n$ for some $n \in \mathscr{Z}$ where $f_n(u) = \hat{u}(n)$.

Now let

$$P_\sigma = \{p_f \mid f \in C^\infty(T)\},\ p_f(u) = |\langle u, f\rangle|,$$

$$P_\beta = \{p_B \mid B \text{ is a } \sigma\text{-bounded subset of } C^\infty(T)\},$$

$$p_B(u) = \sup\{|\langle u, f\rangle| \mid f \in B\},$$

$$P_{\sigma'} = \{p_n \mid n \in \mathscr{Z}\},\ p_n(u) = |\hat{u}(n)|.$$

These are the natural calibrations for σ, β and σ' respectively.

Assertions: For the algebra $D(T)$,

(i) $B_{P_\sigma} = Q_{P_\sigma} = \not{c}\delta$

(ii) $B_{P_\beta} = Q_{P_\beta} = \not{c}\delta$

(iii) $B_{P_{\sigma'}} = \mathrm{PM}(T)$, $Q_{P_{\sigma'}} = D(T)$.

Proof. (i) Let $u \in Q_P$. Then for each $f \in C^\infty(T)$, there is a constant $M_{u,f}$ such that

$$|\langle u * v, f\rangle| \leqslant M_{u,f} |\langle v, f\rangle| \quad (v \in D(T)). \tag{1}$$

Since every distribution on T is of finite order, we can assume that the order of $u \leqslant m$. By [15, § 12.5.7], there is a continuous function ϕ on T such that $u = D^{m+2}\phi + \hat{u}(0)\delta$ where $D^{m+2}\phi$ is the $(n+2)$th distributional derivative of ϕ. Since $u \in Q_{P_\sigma}$, $\delta \in Q_{P_\sigma}$, it follows that $D^{m+2}\phi \in Q_P$. Applying (1) to D^{m+2} with $v = \delta$, we get $|\langle D^{m+2}\phi, f\rangle| \leqslant M_{\phi,f}|f(1)|$ for all $f \in C^\infty(T)$; i.e. $|\int \phi(t) f^{(m+2)}(t)\,dt| \leqslant M_{\phi,f}|f(1)|$ $(f \in C^\infty(T))$. Taking $f(t) = \exp(\mathrm{int}) - 1$ $(n \in \mathscr{Z})$, it follows that $\phi(n) = \int \phi(t) \exp(-\mathrm{int})\,dt = 0$ for all $n \in \mathscr{Z}$. Hence $\phi = 0$ a.e., and by continuity, $\phi(t) = 0$ for all t. Thus $D^{m+2}\phi = 0$, $u = \hat{u}(0)\delta$. Thus $Q_{P_\sigma} = B_{P_\sigma} = \not{c}\delta$. (That $B_{P_\sigma} = \not{c}\delta$ can also alternatively be seen as: for $u \in B_{P_\sigma}$, (1) holds with a constant M_u depending only on u. Since $C^\infty(T)$ is dense in $C(T)$, it follows that $u * v$ is a measure if v is a measure. This forces u to be a measure [15, §12.8.4]. Also, as above, taking $v = \delta$ in (1), $|\int f\,du| \leqslant M_u|f(1)|$ for all $f \in C^\infty(T)$, and so for all $f \in C(T)$. Hence $u = \lambda\delta$ for some $\lambda \in \not{c}$).

(ii) is immediate from (i) in view of $B_{P_\beta} \subset B_{P_\sigma}$, $Q_{P_\beta} \subset Q_{P_\sigma}$. This is because $P_\sigma \subset P_\beta$.

(iii) is a consequence of the submultiplicativity of seminorms in $P_{\sigma'}$.

Remark. More generally, one may look to the structure of distributions on a compact Lie group.

3.9 *Operators on the product of Hilbert spaces*

Let $\{H_\alpha | \alpha \in \Delta\}$ be a family of Hilbert spaces. Let $X = \prod_\alpha X_\alpha$, a locally convex space with calibration $\Gamma = \{p_\alpha | \alpha \in \Delta\}$, $p_\alpha(x) = \|x_\alpha\|$ where $x = (x_\alpha) \in X$. Let $H = \Sigma^{\oplus} H_\alpha = \{x = (x_\alpha) \in X | \Sigma \|x_\alpha\|^2 < \infty\}$ with inner product $\langle x, y \rangle = \Sigma \langle x_\alpha, y_\alpha \rangle$ for $x = (x_\alpha)$, $y = (y_\alpha)$ in H. Let $D = \{x = (x_\alpha) \in X) x_\alpha = 0$ for all but a finite number of $\alpha\}$. The inclusion $H \to X$ is continuous and D is a dense subspace of H and X.

Let $Q(X, \Gamma)$ and $B(X, \Gamma)$ be the algebras consisting of respectively the quotient bounded operators and the universally bounded operators on X. As shown in [9], $(Q(X, \Gamma), t_\Gamma)$ is a b^*-algebra and $(B(X, \Gamma), p_\Gamma)$ is a C^*-algebra. (Recall: The topology t_Γ is determined by the b^*-calibration $\{q_\alpha\}$, $q_\alpha(T) = \sup_\alpha \{p_\alpha(Tx) | p_\alpha(x) \leqslant 1\}$ and $p_\Gamma(T) = \sup_\alpha q_\alpha(T)$). Note $X/\ker p_\alpha$ is identified with H_α with the result, given $T \in Q(X, \Gamma)$, $T^\alpha x_\alpha = (Tx)_\alpha$ is a well-defined bounded linear operator on H_α, $q_\alpha(T) = \|T^\alpha\|$ and the involution on $Q(X, \Gamma)$ is defined by $(T^* x)_\alpha = (T^\alpha)^* x_\alpha$ for $x = (x_\alpha)$ in X.

Now let $L(X)$ be the algebra of all continuous linear operators on (X, Γ). Let $\mathscr{G}$ be either on the following

$\mathscr{G}_1$ = all finite subsets of X,
$\mathscr{G}_2$ = all compact convex subsets of X,
$\mathscr{G}_3$ = all compact subsets of X,
$\mathscr{G}_4$ = all bounded subsets of X.

Let $\mathscr{U}$ be a o-neighbourhood base for X. On $L(X)$, the topology $\tau_{\mathscr{G}}$ of uniform convergence on members of $\mathscr{G}$ is defined by taking a o-neighbourhood base consisting of sets of the form $U(B, V) = \{T \in L(X) | T(B) \subset V\}$ $(B \in \mathscr{G}, V \in \mathscr{U})$. It is given by the calibration $P_{\Gamma, \mathscr{G}} = \{p_{\alpha, B} | \alpha \in \Delta, B \in \mathscr{G}\}$ where $p_{\alpha, B}(T) = \sup\{p_\alpha(Tx) | x \in B\}$. The sets of quotient-bounded and universally-bounded elements of the calibrated locally convex algebra $(L(X), P_{\Gamma, \mathscr{G}})$ are denoted by $Q_{P_{\Gamma, \mathscr{G}}}$ and $B_{P_{\Gamma, \mathscr{G}}}$ respectively. The natural LMC topology $t_{P_{\Gamma, \mathscr{G}}}$ on $Q_{P_{\Gamma, \mathscr{G}}}$ is determined by the seminorms $q_{\alpha, B}(T) = \sup\{p_{\alpha, B}(TS) | p_{\alpha, B}(S) \leqslant 1\}$ and the norm on $B_{P_{\Gamma, \mathscr{G}}}$ is $p(T) \equiv p_{P_{\Gamma, \mathscr{G}}}(T) = \sup_{\alpha, B} q_{\alpha, B}(T)$. Then as in [9, Example 4.1], $Q_{P_{\Gamma, \mathscr{G}}}$ is topologically *isomorphic to $(Q(X, \Gamma), t_\Gamma)$ and $B_{P_{\Gamma, \mathscr{G}}}$ is isometrically *isomorphic to $(B(X, \Gamma), p_\Gamma)$ which in turn is isometrically *isomorphic to a C^*-subalgebra of $\mathscr{B}(H)$.

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References

[1] Allan G R, A spectral theory for locally convex algebras, *Proc. London Math. Soc.* **15** (1965) 399–421
[2] Allan G R, On a class of locally convex algebras, *Proc. London Math. Soc.* **17** (1967) 91–114
[3] Apostal C, b*-algebras and their representations, *J. London Math. Soc.* **3** (1971) 30–38
[4] Bhatt S J, A note on generalized B*-algebras, *J. Indian Math. Soc.* **43** (1979) 253–257
[5] Bhatt S J, A note on generalized B*-algebras–II, *J. Indian Math. Soc.* **48** (1980) 285–290
[6] Bhatt S J, On Arens algebra L^w, *Glas. Mat.* **15** (1980) 305–312
[7] Bhatt S J, On Arens algebra L^w II, *Glas. Mat.* **16** (1981) 297–306
[8] Bhatt S J, On the dual of a generalized B*-algebra, *Math. Rep. Acad. Sci. Canada* **4** (1982) 3–8
[9] Bhatt S J, Quotient bounded elements in locally convex algebras, *J. Aust. Math. Soc.* **A33** (1982) 102–113
[10] Bonsall F F and Duncan J, *Complete normed algebras*, Springer-Verlag, Berlin, Heidelberg, New York (1973)
[11] Brooks R M, On locally m-convex* algebras, *Pacific J. Math.* **25** (1967) 5–23
[12] Brooks R M, On representing F* algebras, *Pacific J. Math.* **39** (1971) 51–69
[13] Dales H G, Automatic continuity: a survey, *Bull. London Math. Soc.* **10** (1978) 129–182
[14] Dixen P G, Generalized B*-algebras, *Proc. London Math. Soc.* **23** (1970) 639–715
[15] Edwards R E, *Fourier series II*, Hoht, Rinehart and Winston (1967)
[16] Giles J R and Koehler D O, On numerical ranges of elements of locally m-convex algebras, *Pacific J. Math.* **59** (1973) 79–91
[17] Giles J R, Koehler D O, Joseph G and Sims B, On numerical ranges of operators on locally convex spaces, *J. Aust. Math. Soc.* **20** (1975) 468–482
[18] Gillman L and Jerison M, Rings of continuous functions, van Nostrand, New York (1960)
[19] Inoue A, Locally C*-algebras, *Mem. Pac. Sci. Kyushu Univ.* **A26** (1971) 197–235
[20] Inoue A and Kuriyalma K, Weakly unbounded operator algebras, *Mem. Pac. Sci. Kyushu Univ.* **31** (1977) 267–284
[21] Inoue A, On a class of unbounded operator algebras I, *Pacific J. Math.* **65** (1976) 36–57
[22] Inoue A, On a class of unbounded operator algebras II, *Pacific J. Math.* **66** (1976) 411–431
[23] Inoue A, On a class of unbounded operator algebras III, *Pacific J. Math.* **67** (1977) 105–115
[24] Inoue A, Representations of GB*-algebras I, *Fukuyoka Univ. Sci. Report* **5** (1975) 63–78
[25] Michael E A, Locally multiplicatively convex topological algebras, *Mem. Am. Math. Soc.* **10** (1952)
[26] Powers R T, Self-adjoint algebras of unbounded operators, *Comm. Math. Phys.* **22** (1971) 88–124
[27] Rickart C G, *General theory of Banach algebras*, van Nostrand, New York (1960)
[28] Rudin W, *Functional analysis*, McGraw Hill, New York (1973)
[29] Saito S, On noncommutative hardy spaces associated with flows on finits von Neumann algebras, *Tohoku Math. J.* **29** (1977) 585–595
[30] Sakai S, *C*- and W*-algebras*, Springler-Verlag, Berlin, Heidelberg, New York (1971)
[31] Schaeffer H H, *Topological vector spaces*, Macmillan, New York (1966)
[32] Treves F, *Topological vector spaces, Distribution and Kernels*, Academic Press, New York (1967)
[33] Wilde I F, The free Fermion field as a Markov field, *J. Functional Anal.* **15** (1974) 12–32
[34] Wood A W, Numerical ranges and generalized B*-algebras, *Proc. London Math. Soc.* **34** (1977) 245–268
[35] Yeadon F J, Noncommutative L^p-spaces, *Math. Proc. Cambridge Phil. Soc.* **77** (1975) 91–102

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Rayleigh wave scattering due to a rigid barrier in a liquid halfspace

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Abstract. The paper presents a theoretical formulation for studying scattering of Rayleigh waves due to the presence of rigid barriers in oceanic waters. The Wiener-Hopf technique has been employed to solve the problem. Exact solution has been obtained in terms of Fourier integrals whose evaluation gives the reflected, transmitted and scattered waves. The scattered waves have the behaviour of cylindrical waves originating at the edge of the barrier. Numerical results for the amplitude of the scattered waves have been obtained for small depth of the barrier.

Keywords. Rayleigh wave scattering; rigid barrier; liquid halfspace; Wiener-Hopf technique.

1. Introduction

We propose to study the problem of scattering of Rayleigh waves due to rigid barriers in the surface of oceanic waters. Ursell [8] considered the effect of a vertical barrier of finite depth, fixed in an infinitely deep sea, on normally incident surface waves. Faulkner [4] extended his solution to a three-dimensional case to study diffraction of an obliquely incident surface wave by a vertical barrier of finite depth. Deshwal [1, 2] studied problems of diffraction of compressional waves due to rigid barriers in a liquid halfspace and a liquid layer. The rigid barrier permits no displacement across it. The method of solution is the Wiener-Hopf technique [5].

2. Statement of the problem

The rigid barrier is held in the free surface of sea water, taken to be a homogenous isotropic liquid halfspace. The halfspace lies to the side of positive z-axis pointing vertically downwards with x-axis in the free surface. The barrier is taken along the z-axis and is of small depth H. A time harmonic two-dimensional Rayleigh wave is incident at the barrier in figure 1.

The two-dimensional wave equation is

$$\frac{\partial^2\bar{\phi}}{\partial x^2}+\frac{\partial^2\bar{\phi}}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2\bar{\phi}}{\partial t^2}+\frac{\varepsilon}{c^2}\frac{\partial\bar{\phi}}{\partial t}, \tag{1}$$

c is the velocity of Rayleigh waves and ε is small positive damping constant. Let the

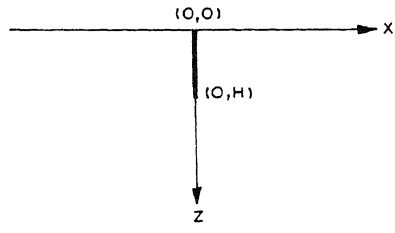

Figure 1.

potential for a time-harmonic wave be

$$\bar{\phi}(x, z, t) = \phi(x, z) \exp(-i\omega t). \tag{2}$$

Then equation (1) reduces to

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} + k^2 \phi = 0 \tag{3}$$

where

$$k^2 = (\omega^2 + i\varepsilon\omega)/c^2, \quad k = k_1 + ik_2. \tag{4}$$

The imaginary part of k is small and positive. Let the incident and total potentials be

$$\phi_i(x, z) = A_0 \exp(-i\alpha_0 x) \exp(-\gamma_0 z), \tag{5}$$

$$\phi_t(x, z) = \phi_i(x, z) + \phi(x, z), \tag{6}$$

α_0 is the root of the Rayleigh equation and $\gamma_0 = (\alpha_0^2 - k^2)^{1/2}$. The displacement components (u, w) at any point (x, z) are given by

$$u = \partial\phi_t/\partial x, \quad w = \partial\phi_t/\partial z \tag{7}$$

3. Boundary conditions

The conditions on the boundaries are

(i) $\phi(x, z)$ is bounded when $z \to \infty$, (8)

(ii) $\phi_t(x, z) = 0$, on $z = 0$, for all x, (9)

(iii) $u = \partial\phi_t/\partial x = 0$, on $x = 0$, $0 \leqslant z \leqslant H$. (10)

We assume that for given z

$$\left.\begin{aligned} |\phi(x, z)| &\sim D_1 \exp(-k_2 x) \text{ as } x \to \infty \\ &\sim D_2 \exp(k_2 x) \text{ as } x \to -\infty. \end{aligned}\right\} \tag{11}$$

where D_1 and D_2 are constants, so that we can define the Fourier transforms

$$\bar{\phi}(\alpha, z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \phi(x, z) \exp(i\alpha x)\, dx, \tag{12}$$

$$\bar{\phi}_+(\alpha, z) = \frac{1}{(2\pi)^{1/2}} \int_{0}^{\infty} \phi(x, z) \exp(i\alpha x)\, dx, \tag{13}$$

$$\bar{\phi}_-(\alpha, z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{0} \phi(x, z) \exp(i\alpha x)\, dx. \tag{14}$$

Let us further take $\alpha = \sigma + i\tau$, then (13) gives

$$|\bar{\phi}_+(\alpha, z)| \leqslant \frac{1}{(2\pi)^{1/2}} \int_{0}^{\infty} \phi(x, z) \exp(-\tau x)\, dx. \tag{15}$$

On account of the assumption made for the form of $\phi(x, z)$, $\bar{\phi}(\alpha, z)$ is bounded at $x = \infty$, only when $\tau > -k_2$. Hence $\bar{\phi}_+(\alpha, z)$ is regular in the region $\tau > -k_2$ of the complex α-plane. Similarly, it can be shown that $\bar{\phi}_-(\alpha, z)$ is regular in the region $\tau < k_2$ of the α-plane. Since

$$\bar{\phi}(\alpha, z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \phi(x, z) \exp(i\alpha x)\, dx = \bar{\phi}_+(\alpha, z) + \bar{\phi}_-(\alpha, z). \tag{16}$$

Therefore

$$|\bar{\phi}| \leqslant |\bar{\phi}_+| + |\bar{\phi}_-| \tag{17}$$

shows that $\bar{\phi}(\alpha, z)$ is regular in the strip $-k_2 < \tau < k_2$ of the α-plane. Similarly derivatives of $\bar{\phi}(\alpha, z)$ are also regular in the same strip.

4. Solution of the problem

Multiplying (3) by $\exp(i\alpha x)/(2\pi)^{1/2}$ and integrating between $x = -\infty$ and $x = \infty$, noting that $\partial\phi/\partial x$ vanishes at $x = \pm\infty$, we have

$$(d^2\bar{\phi}/dz^2) - \gamma^2\bar{\phi} = 0, \tag{18}$$

where

$$\gamma = \pm(\alpha^2 - k^2)^{1/2}. \tag{19}$$

We choose that sign before the radical which makes the real part of $\gamma \geqslant 0$, for all α. The solution of (18) is

$$\bar{\phi}(\alpha, z) = A(\alpha) \exp(-\gamma z) + B(\alpha) \exp(\gamma z). \tag{20}$$

Applying boundary condition that $\phi(x, z)$ and hence $\bar{\phi}(\alpha, z)$ is bounded when z tends to ∞, we must take in (20)

$$B(\alpha) = 0. \tag{21}$$

Hence

$$\bar{\phi}(\alpha, z) = A(\alpha) \exp(-\gamma z). \tag{22}$$

This can be written as

$$\bar{\phi}_+(\alpha, z) + \bar{\phi}_-(\alpha, z) = A(\alpha)\exp(-\gamma z). \tag{23}$$

Putting $z = H$ in (23) and in its derivative with respect to z, and eliminating $A(\alpha)$ between the equations so obtained, we have

$$\bar{\phi}_+(\alpha) + \bar{\phi}_-(\alpha) = -\gamma^{-1}[\bar{\phi}'_+(\alpha) + \bar{\phi}'_-(\alpha)]. \tag{24}$$

We have used $\bar{\phi}_+(\alpha)$, $\bar{\phi}_-(\alpha)$ for $\bar{\phi}_+(\alpha, H)$ and $\bar{\phi}_-(\alpha, H)$. Similar notions are used for their derivatives. Changing α to $-\alpha$ in (24), we get

$$\bar{\phi}_+(-\alpha) + \bar{\phi}_-(-\alpha) = -\bar{\gamma}^1[\bar{\phi}'_+(-\alpha) + \bar{\phi}'_-(-\alpha)]. \tag{25}$$

Adding (24) and (25), we have

$$\bar{\phi}_+(\alpha) + \bar{\phi}_-(\alpha) + \bar{\phi}_+(-\alpha) + \bar{\phi}_-(-\alpha)$$

$$= -\bar{\gamma}^1[\bar{\phi}'_+(\alpha) + \bar{\phi}'_-(\alpha) + \bar{\phi}'_+(-\alpha) + \bar{\phi}'_-(-\alpha)]. \tag{26}$$

Again multiplying (3) by $\exp(i\alpha x)/(2\pi)^{1/2}$ and integrating with respect to x, as x varies from 0 to ∞, we get

$$\frac{d^2\bar{\phi}_+(\alpha, z)}{dz^2} - \gamma^2\bar{\phi}_+(\alpha, z) = \frac{1}{(2\pi)^{1/2}}\left(\frac{\partial\phi}{\partial x}\right)_0 - \frac{i\alpha}{(2\pi)^{1/2}}(\phi)_0. \tag{27}$$

Changing α to $-\alpha$ in (27) and adding the two equations, we find

$$\frac{d^2}{dz^2}[\bar{\phi}_+(\alpha, z) + \bar{\phi}_+(-\alpha, z)] - \gamma^2[\bar{\phi}_+(\alpha, z) + \bar{\phi}_+(-\alpha, z)]$$

$$= \frac{2}{(2\pi)^{1/2}}(\partial\phi/\partial x)_0, \tag{28}$$

where $(\partial\phi/\partial x)_0$ means value of $\partial\phi/\partial x$ at $x = 0$. To find the value of $(\partial\phi/\partial x)_0$, we use condition (10) to find out

$$(\partial\phi/\partial x)_0 = i\alpha_0 A_0 \exp(-\gamma_0 z). \tag{29}$$

Using this value in (28) we have

$$\frac{d^2}{dz^2}[\bar{\phi}_+(\alpha, z) + \bar{\phi}_+(-\alpha, z)] - \gamma^2[\bar{\phi}_+(\alpha, z) + \bar{\phi}_+(-\alpha, z)]$$

$$= \frac{2i\alpha_0 A_0 \exp(-\gamma_0 z)}{(2\pi)^{1/2}}. \tag{30}$$

The complete solution of (30) is given by

$$\bar{\phi}_+(\alpha, z) + \phi(-\alpha, z) = A_1(\alpha)\exp(\gamma z) + A_2(\alpha)\exp(-\gamma z)$$

$$- \frac{2}{(2\pi)^{1/2}}\frac{i\alpha_0 A_0 \exp(-\gamma_0 z)}{(\alpha^2 - \alpha_0^2)}. \tag{31}$$

To eliminate the arbitrary constants $A_2(\alpha)$ and $A_1(\alpha)$, we use (9) to find

$$(\phi(x, z))_{z=0} = -A_0 \exp(-i\alpha_0 x). \tag{32}$$

Using (32) in (13), we get

$$\bar{\phi}_+(\alpha, 0) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty (\phi)_{z=0} \exp(i\alpha x)\, dx = \frac{-A_0}{(2\pi)^{1/2}} \left| \frac{\exp[i(\alpha-\alpha_0)x]}{i(\alpha-\alpha_0)} \right|_0^\infty. \tag{33}$$

The integral in (33) is convergent on the line $\tau = \mathrm{Im}(\alpha_0)$, except at the point $\alpha = \alpha_0$. Hence from (33), we have

$$\bar{\phi}_+(\alpha, 0) = \frac{-iA_0}{(2\pi)^{1/2}(\alpha-\alpha_0)}. \tag{34}$$

Changing α to $-\alpha$ in (34), we have

$$\bar{\phi}_+(-\alpha, 0) = \frac{iA_0}{(2\pi)^{1/2}(\alpha+\alpha_0)}. \tag{35}$$

Adding (34) and (35), we find that

$$\bar{\phi}_+(\alpha, 0) + \bar{\phi}_+(-\alpha, 0) = \frac{-2i\alpha_0 A_0}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)}. \tag{36}$$

Again putting $z = 0$ in (31) and using (36), we see that

$$A_1(\alpha) + A_2(\alpha) = 0, \text{ i.e., } A_2(\alpha) = -A_1(\alpha). \tag{37}$$

The solution of (31) is now

$$\bar{\phi}_+(\alpha, z) + \bar{\phi}_+(-\alpha, z) = 2A_1 \sinh \gamma z - \frac{2}{(2\pi)^{1/2}} \frac{i\alpha_0 A_0 \exp(-\gamma_0 z)}{(\alpha^2-\alpha_0^2)}. \tag{38}$$

Putting $z = H$ in (38), we get

$$\bar{\phi}_+(\alpha) + \bar{\phi}_+(-\alpha) = 2A_1 \sinh \gamma H - \frac{2i\alpha_0 A_0 \exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)}. \tag{39}$$

Differentiating (38) with respect to z and putting $z = H$, we have

$$\bar{\phi}'_+(\alpha) + \bar{\phi}'_+(-\alpha) = 2A_1\gamma \cosh \gamma H + \frac{2i\alpha_0\gamma_0 A_0 \exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)}. \tag{40}$$

Eliminating $A_1(\alpha)$ between (39) and (40), we get

$$\bar{\phi}_+(\alpha) + \bar{\phi}_+(-\alpha) = \bar{\gamma}^1 \tanh \gamma H [\bar{\phi}_+(\alpha) + \bar{\phi}'_+(-\alpha)]$$

$$- \frac{2i\alpha_0 A_0 \exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)} - \frac{2i\alpha_0\gamma_0 A_0}{(2\pi)^{1/2}\gamma(\alpha^2-\alpha_0^2)} \tanh \gamma H \exp(-\gamma_0 H). \tag{41}$$

Again multiplying (3) by $\exp(i\alpha x)/(2\pi)^{1/2}$ and integrating with respect to x, between the

limits $-\infty$ to 0, we have

$$\frac{d^2}{dz^2}[\bar{\phi}_-(\alpha, z)] - \gamma^2[\bar{\phi}_-(\alpha, z)] = \frac{-1}{(2\pi)^{1/2}}\left(\frac{\partial\phi}{\partial x}\right)_{x=0} + \frac{i\alpha}{(2\pi)^{1/2}}(\phi)_{x=0}. \quad (42)$$

Replacing α by $-\alpha$ in (42) and adding it to the resulting equation, it is found that

$$\frac{d^2}{dz^2}[\bar{\phi}_-(\alpha, z) + \bar{\phi}_-(-\alpha, z)] - \gamma^2[\bar{\phi}_-(\alpha, z) + \bar{\phi}_-(-\alpha, z)]$$

$$= \frac{-2}{(2\pi)^{1/2}}\left(\frac{\partial\phi}{\partial x}\right)_{x=0}. \quad (43)$$

Using (29) in (42), we get

$$\frac{d^2}{dz^2}[\bar{\phi}_-(\alpha, z) + \bar{\phi}_-(-\alpha, z)] - \gamma^2[\bar{\phi}_-(\alpha, z) + \bar{\phi}_-(-\alpha, z)]$$

$$= \frac{-2}{(2\pi)^{1/2}} i\alpha_0 A_0 \exp(-\gamma_0 z). \quad (44)$$

The complete solution of (44) is given by

$$\bar{\phi}_-(\alpha, z) + \bar{\phi}_-(-\alpha, z) = \beta_1(\alpha)\exp(\gamma z) + \beta_2(\alpha)\exp(-\gamma z) + \frac{2}{(2\pi)^{1/2}}\frac{i\alpha_0 A_0 \exp(-\gamma_0 z)}{\alpha^2 - \alpha_0^2}. \quad (45)$$

To eliminate $\beta_1(\alpha)$ and $\beta_2(\alpha)$, we use (32) in

$$\bar{\phi}_-(\alpha, 0) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{0}(\phi(x, z))_{z=0}\exp(i\alpha x)\,dx, \quad (46)$$

to get

$$\bar{\phi}_-(\alpha, 0) = \frac{-A_0}{(2\pi)^{1/2}}\left|\frac{\exp[i(\alpha-\alpha_0)x]}{i(\alpha-\alpha_0)}\right|_{-\infty}^{0}. \quad (47)$$

The integral in (47) is convergent on the line $\tau = \mathrm{Im}(\alpha_0)$, except the point $\alpha = \alpha_0$, and we find

$$\bar{\phi}_-(\alpha, 0) = \frac{iA_0}{(2\pi)^{1/2}(\alpha-\alpha_0)}. \quad (48)$$

Changing α to $-\alpha$ in (48) and adding the resulting equation to (48), wet get

$$\bar{\phi}_-(\alpha, 0) + \bar{\phi}_-(-\alpha, 0) = \frac{2i\alpha_0 A_0}{(2\pi)^{1/2}(\alpha^2 - \alpha_0^2)}. \quad (49)$$

Putting $z = H$ in (45) and using (49), we find

$$\beta_1(\alpha) + \beta_2(\alpha) = 0, \text{ i.e., } \beta_2(\alpha) = -\beta_1(\alpha). \quad (50)$$

Using (45) and (50), we obtain

$$\bar{\phi}_-(\alpha, z) + \bar{\phi}_-(-\alpha, z) = 2\beta_1 \sinh\gamma z + \frac{2}{(2\pi)^{1/2}}\frac{i\alpha_0 A_0 \exp(-\gamma_0 z)}{(\alpha^2 - \alpha_0^2)}. \quad (51)$$

Putting $z = H$ in (51) and in its derivative with respect to z, we get the following equations

$$\bar{\phi}_-(\alpha) + \bar{\phi}_-(-\alpha) = 2\beta_1 \sinh \gamma H + \frac{2}{(2\pi)^{1/2}} \frac{i\alpha_0 A_0 \exp(-\gamma_0 H)}{(\alpha^2 - \alpha_0^2)}, \tag{52}$$

$$\bar{\phi}'_-(\alpha) + \bar{\phi}'_-(-\alpha) = 2\beta_1 \gamma \cosh \gamma H - \frac{2i\alpha_0 \gamma_0 A_0 \exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2 - \alpha_0^2)}. \tag{53}$$

Eliminating β_1, between (52) and (53), we have

$$\bar{\phi}_-(\alpha) + \bar{\phi}_-(-\alpha) = \bar{\gamma}^1 \tanh \gamma H [\bar{\phi}'_-(\alpha) + \bar{\phi}'_-(-\alpha)] +$$

$$\frac{2}{(2\pi)^{1/2}} \frac{i\alpha_0 A_0 \exp(-\gamma_0 H)}{(\alpha^2 - \alpha_0^2)} + \frac{2}{(2\pi)^{1/2}} \frac{i\alpha_0 \gamma_0 A_0 \exp(-\gamma_0 H)}{(\alpha^2 - \alpha_0^2)\gamma} \tanh \gamma H. \tag{54}$$

Adding (54) and (41), we obtain

$$\bar{\phi}_+(\alpha) + \bar{\phi}_+(-\alpha) + \bar{\phi}_-(\alpha) + \bar{\phi}_-(-\alpha) = \bar{\gamma}^1 \tanh \gamma H [\bar{\phi}'_+(\alpha) + \bar{\phi}'_+(-\alpha) + \bar{\phi}'_-(\alpha) + \bar{\phi}_-(-\alpha)] \tag{55}$$

From (55) and (26), it can be easily seen that

$$(1 + \tanh \gamma H)[\bar{\phi}_+(\alpha) + \bar{\phi}_+(-\alpha) + \bar{\phi}_-(\alpha) + \bar{\phi}_-(-\alpha)] = 0. \tag{56}$$

Since $(1 + \tanh \gamma H) \neq 0$, therefore, we can write (56) as

$$\bar{\phi}_+(\alpha) + \bar{\phi}_-(-\alpha) = -\bar{\phi}_+(-\alpha) - \bar{\phi}_-(\alpha). \tag{57}$$

This is a functional equation of Wiener-Hopf type defined in the strip $-k_2 < \tau < k_2$. The function $\bar{\phi}_+(\alpha) + \bar{\phi}_-(-\alpha)$ is analytic in the region $\tau > -k_2$ and $-\bar{\phi}_-(\alpha) - \bar{\phi}_+(-\alpha)$ in the region $\tau < k_2$ of the complex α-plane. The two functions have the strip $-k_2 < \tau < k_2$ as common region of analyticity, in which the two functions are identically equal to each other. Therefore, by analytic continuation, the two functions, in their respective regions of analyticity, are the representatives of an entire function $F(\alpha)$. From (13), we see that

$$|\bar{\phi}_+(\alpha, z)| \leqslant \frac{1}{(2\pi)^{1/2}} \int_0^\infty |\phi(x, z)| \exp(-\tau x)\, dx. \tag{58}$$

$\bar{\phi}_+(\alpha, z)$ tends to zero as $|\alpha|$ tends to ∞. Similarly $\bar{\phi}_+(-\alpha)$, $\bar{\phi}_-(\alpha)$ and $\bar{\phi}_-(-\alpha)$ tend to zero as $|\alpha|$ tends to ∞. Hence by Liouville's theorem, $F(\alpha)$ is zero at all points of the α-plane. Equating to zero, each member of (57) to find out

$$\bar{\phi}_+(\alpha) = -\bar{\phi}_-(-\alpha), \quad \bar{\phi}_+(-\alpha) = -\bar{\phi}_-(\alpha).$$

Similarly

$$\bar{\phi}'_+(\alpha) = -\bar{\phi}'_-(-\alpha), \quad \bar{\phi}'_+(-\alpha) = -\bar{\phi}'_-(\alpha). \tag{59}$$

Subtracting (24) from (41) and using (59), we get

$$2\overline{\phi}_+(-\alpha) = \overline{\gamma}^1(1+\tanh\gamma H)\,\overline{\phi}'_+(\alpha) + \overline{\gamma}^1(\tanh\gamma H - 1)\overline{\phi}'_+(-\alpha)$$

$$-\frac{2}{(2\pi)^{1/2}}\frac{i\alpha_0 A_0 \exp(-\gamma_0 H)}{(\alpha^2-\alpha_0^2)} - \frac{2i\alpha_0\gamma_0 A_0 \tanh\gamma H \exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)\gamma}$$

$$= \frac{\exp(\gamma H)}{\gamma\cosh\gamma H}\overline{\phi}'_+(\alpha) - \frac{\exp(-\gamma H)\overline{\phi}'_+(-\alpha)}{\gamma\cosh\gamma H}$$

$$-\frac{2i\alpha_0\gamma_0 A_0 \exp(-\gamma_0 H)\exp(\gamma H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)\gamma\ \cosh\gamma H}$$

$$+\frac{2i\alpha_0\gamma_0 A_0 \exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)\gamma} - \frac{2i\alpha_0 A_0 \exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)},$$

i.e.

$$2\overline{\phi}_+(-\alpha) = \frac{\overline{\phi}'_+(\alpha)}{(\alpha^2-k^2)H}\left(\frac{\exp(\gamma H)\cdot\gamma H}{\cosh\gamma H}\right) - \frac{\overline{\phi}'_+(-\alpha)}{(\alpha^2-k^2)H}\left(\frac{\exp(-\gamma H)\cdot\gamma H}{\cosh\gamma H}\right)$$

$$-\frac{2i\alpha_0\gamma_0 A_0 \exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2-k^2)H(\alpha^2-\alpha_0^2)}\left(\frac{\exp(\gamma H)\cdot\gamma H}{\cosh\gamma H}\right)$$

$$+\frac{2i\alpha_0\gamma_0 A_0 \exp(-\gamma_0 H)}{(2\pi)^{1/2}\gamma(\alpha^2-\alpha_0^2)} - \frac{2i\alpha_0 A_0 \exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)}. \tag{60}$$

This differential equation is solved by the Wiener-Hopf technique.

Let us now factorize $\cosh\gamma H \exp(-\gamma H)/\gamma H$. We write

$$\exp(-\gamma H) = \exp[-T_+(\alpha) - T_-(\alpha)], \tag{61}$$

where

$$T_+(\alpha) = \gamma H\pi^{-1}\cos^{-1}(\alpha/k) \sim i\alpha H\pi^{-1}\log(2\alpha/k), \text{ as } \alpha\to\infty, \tag{62}$$

and

$$T_-(\alpha) = T_+(-\alpha). \tag{63}$$

The factorization of $\cosh\gamma H \exp(-\gamma H)/\gamma H$ as an infinite product is

$$L(\alpha) = L_+(\alpha)L_-(\alpha) = [\exp(-\gamma H)\cosh\gamma H]/\gamma H$$

$$= \frac{\exp[-T_+(\alpha)-T_-(\alpha)]}{H(\alpha+k)^{1/2}(\alpha-k)^{1/2}}\prod_{n=1}^{\infty}[1-k^2b^2{}_{n-1/2}+\alpha^2b^2{}_{n-1/2}], \tag{64}$$

where

$$L_-(\alpha) = \frac{\exp[x(\alpha)-T_-(\alpha)]}{[H(\alpha-k)]^{1/2}}\prod_{n=1}^{\infty}[(1-k^2b_{n-1/2}^2)^{1/2}+i\alpha b_{n-1/2}]\cdot$$

$$\exp(-i\alpha b_{n-1/2}), \tag{65}$$

$b_{n-1/2} = H/(n-1/2)\pi$ and $x(\alpha)$ is an arbitrary function to make a suitable behaviour of

$L_-(\alpha)$ as $|\alpha| \to \infty$. The behaviour of $L_-(\alpha)$ as $|\alpha| \to \infty$ is given by

$$L_-(\alpha) \sim \exp \frac{\exp[x(\alpha) + i\alpha H\pi^{-1} \log(-2\alpha/k)]}{[H(\alpha - k)]^{1/2}}$$

$$\prod_{n=1}^{\infty} \left[1 + \frac{i\alpha H\pi^{-1}}{n-1/2}\right] \exp\left(\frac{-i\alpha H\pi^{-1}}{n-1/2}\right). \tag{66}$$

The infinite product in (66) is obtained using the result (Noble, [5] ex 1.10, p. 41)

$$\prod_{n=1}^{\infty} \left[1 + \frac{\alpha}{n-1/2}\right] \exp\left(\frac{-\alpha}{n-1/2}\right) \sim \exp(-c_1\alpha) 2^{-2\alpha} \exp(\alpha + 1/2)\alpha^{-\alpha}, \tag{67}$$

where $c_1 = 0.5772$ is Euler's constant. Therefore

$$L_-(\alpha) \sim \exp[x(\alpha) + i\alpha H\pi^{-1}(1 - c_1 + \log(\pi/2kH)) - \alpha H/2]/[\pi(\alpha - k)]^{1/2}, \tag{68}$$

$L_-(\alpha)$ is asymptotic to $1/(\alpha)^{1/2}$ as $|\alpha| \to \infty$ if we choose

$$x(\alpha) = -i\alpha H\pi^{-1}[1 - c_1 + \log(\pi/2kH)] + \alpha H/2. \tag{69}$$

Substituting the value of $\exp(-\gamma H)\cosh \gamma H/\gamma H$ in (60), we have

$$2\overline{\phi}_+(-\alpha) = \frac{\overline{\phi}'_+(\alpha)}{(\alpha^2 - k^2)HL_+(\alpha)L_-(\alpha)} - \frac{\overline{\phi}'_+(-\alpha)\exp(-2\gamma H)}{(\alpha^2 - k^2)HL_+(\alpha)L_-(\alpha)}$$

$$- \frac{2i\alpha_0 A_0 \gamma_0 \exp(-\gamma_0 H)}{(2\pi)^{1/2} H(\alpha^2 - k^2)(\alpha^2 - \alpha_0^2)L_+(\alpha)L_-(\alpha)}$$

$$+ \frac{2i\alpha_0\gamma_0 A_0 \exp(-\gamma_0 H)}{(\alpha^2 - k^2)^{1/2}(2\pi)^{1/2}(\alpha^2 - \alpha_0^2)}$$

$$- \frac{2}{(2\pi)^{1/2}} \frac{i\alpha_0 A_0 \exp(-\gamma_0 H)}{(\alpha^2 - \alpha_0^2)}. \tag{70}$$

This can be written as

$$2\overline{\phi}_+(-\alpha)(\alpha - k)L_-(\alpha) + \frac{\overline{\phi}'_+(-\alpha)\exp(-2\gamma H)}{H(\alpha + k)L_+(\alpha)}$$

$$- \frac{2i\alpha_0\gamma_0 A_0 \exp(-\gamma_0 H)(\alpha - k)^{1/2} \cdot L(-\alpha)}{(\alpha + k)^{1/2} \cdot (2\pi)^{1/2}(\alpha^2 - \alpha_0^2)}$$

$$+ \frac{2i\alpha_0 A_0 (\alpha - k) L_-(\alpha) \exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2 - \alpha_0^2)}$$

$$= \frac{\overline{\phi}'_+(\alpha)}{H(\alpha + k)L_+(\alpha)} - \frac{2i\alpha_0 A_0 \gamma_0 \exp(-\gamma_0 H)}{(2\pi)^{1/2} H(\alpha + k)(\alpha^2 - \alpha_0^2)L_+(\alpha)}. \tag{71}$$

The term $\overline{\phi}'_+(-\alpha)\exp(-2\gamma H)/H(\alpha + k)\ L_+(\alpha)$ in (71) is analytic inside the strip $-k_2 < \tau < k_2$ and we have shown that $L_+(\alpha)$ behaves as $1/(\alpha)^{1/2}$ as $|\alpha|$ tends to ∞, inside the said region, the above mentioned term tends to zero. The term under

consideration can be expressed as sum of two functions one of which is analytic in the region $\tau < k_2$ and other is analytic in the region $\tau > -k_2$. Terms in (71) are decomposed as

$$
\begin{aligned}
&2\overline{\phi}_+(-\alpha)(\alpha-k)L_-(\alpha)+\frac{1}{H(\alpha+k)L_+(\alpha)}\left[\overline{\phi}'_+(-\alpha)\exp(-2\gamma H)\right.\\
&\left.-\overline{\phi}'_+(-p_n)\exp\left[-2H(p_n^2-k^2)^{1/2}\right]\right.\\
&\quad-\frac{2i\alpha_0 A_0\gamma_0\exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha+\alpha_0)(\alpha+k)^{1/2}}\left[\frac{(\alpha-k)^{1/2}\cdot L_-(\alpha)}{\alpha-\alpha_0}-\frac{(p_i-k)^{1/2}\ L_-(p_i)}{p_i-\alpha_0}\right]\\
&\quad+\frac{2i\alpha_0 A_0\exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha+\alpha_0)}\left[\frac{(\alpha-k)L_-(\alpha)}{\alpha-\alpha_0}-\frac{(\alpha_0+k)L_-(-\alpha_0)}{2\alpha_0}\right]\\
&\quad+\frac{i\gamma_0 A_0\exp(-\gamma_0 H)}{(2\pi)^{1/2}\ H(\alpha-\alpha_0)(\alpha_0+k)L_+(\alpha_0)}\\
&=\frac{\overline{\phi}'_+(\alpha)}{H(\alpha+k)L_+(\alpha)}-\frac{\overline{\phi}'_+(-p_n)\exp(-2H(p_n^2-k^2)^{1/2})}{H(\alpha+k)L_+(\alpha)}\\
&+\frac{2i\alpha_0\gamma_0 A_0(p_i-k)^{1/2}L_-(p_i)\exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha+\alpha_0)(\alpha+k)^{1/2}(p_i-\alpha_0)}\\
&\quad-\frac{iA_0(\alpha_0+k)L_-(-\alpha_0)\exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha+\alpha_0)}-\frac{2i\alpha_0\gamma_0 A_0\exp(-\gamma_0 \mathrm{H})}{(2\pi)^{1/2}H(\alpha-\alpha_0)}\\
&\quad\times\left[\frac{1}{(\alpha+k)(\alpha+\alpha_0)L_+(\alpha)}-\frac{1}{2\alpha_0(\alpha_0+k)L_+(\alpha_0)}\right],
\end{aligned}
\tag{72}
$$

where p_n are roots of $(\alpha+k)L_+(\alpha)=0$ and $p_i=-\alpha_0, -k$. The functions on the left and right sides of (72) are analytic in the regions $\tau < k_2$ and $\tau > -k_2$ respectively of the α plane. Hence, by analytic continuation, they represent an entire function $R(\alpha)$. Moreover $L_\pm(\alpha)$ reduce to $1/(\alpha)^{1/2}$ as $|\alpha|$ tends to ∞, each term on the right side tends to zero as $\alpha\to\infty$. Hence, by Liouville's theorem, $R(\alpha)$ is zero at every point of the complex plane. Equating to zero the right side of (72), we get

$$
\begin{aligned}
\overline{\phi}'_+(\alpha)=\ &\overline{\phi}'_+(-p_n)\exp\left[-2H(p_n^2-k^2)^{1/2}\right]\\
&-\frac{2i\alpha_0\gamma_0 HA_0(\alpha+k)^{1/2}(p_i-k)^{1/2}\ L_-(p_i)L_+(\alpha)\exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha+\alpha_0)(p_i-\alpha_0)}\\
&+\frac{iH(\alpha+k)(\alpha_0+k)A_0L_+(\alpha)L_-(-\alpha_0)\exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha+\alpha_0)}\\
&+\frac{2i\alpha_0\gamma_0 A_0\exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)}\\
&-\frac{i\gamma_0(\alpha+k)A_0L_+(\alpha)\exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha-\alpha_0)(\alpha_0+k)L_+(\alpha_0)}.
\end{aligned}
\tag{73}
$$

Changing α to $-\alpha$ in (73) and subtracting the resulting equation, we find

$$\overline{\phi}'_+(\alpha)-\overline{\phi}'_+(-\alpha)=\overline{\phi}'_+(\alpha)+\overline{\phi}'_-(\alpha)=\overline{\phi}'(\alpha)$$

$$=-\frac{2i\alpha_0\gamma_0 A_0 H(p_i-k)^{1/2}L_-(p_i)\exp(-\gamma_0 H)}{(2\pi)^{1/2}(p_i-\alpha_0)}$$

$$\cdot\left[\frac{(\alpha+k)^{1/2}L_+(\alpha)}{\alpha+\alpha_0}+\frac{(k-\alpha)^{1/2}L_+(-\alpha)}{\alpha-\alpha_0}\right]$$

$$+\frac{iHA_0(\alpha_0+k)L_-(-\alpha_0)\exp(-\gamma_0 H)}{(2\pi)^{1/2}}\left[\frac{(\alpha+k)L_+(\alpha)}{\alpha+\alpha_0}+\frac{(k-\alpha)L_+(-\alpha)}{\alpha-\alpha_0}\right]$$

$$-\frac{i\gamma_0 A_0\exp(-\gamma_0 H)}{(2\pi)^{1/2}(\alpha_0+k)L_+(\alpha_0)}\left[\frac{(\alpha+k)L_+(\alpha)}{\alpha-\alpha_0}+\frac{(k-\alpha)L_+(-\alpha)}{\alpha+\alpha_0}\right]. \tag{74}$$

Differentiating (23) with respect to z and putting $z=H$, we get

$$\overline{\phi}'_+(\alpha)+\overline{\phi}'_-(\alpha)=\overline{\phi}'(\alpha)=-\gamma A(\alpha)\exp(-\gamma H) \tag{75}$$

From here

$$A(\alpha)=-\frac{1}{\gamma}[\overline{\phi}'_+(\alpha)+\overline{\phi}'_-(\alpha)]\exp(\gamma H). \tag{76}$$

Using (76) and (74), we find

$$A(\alpha)=\frac{2i\alpha_0\gamma_0 HA_0(p_i-k)^{1/2}\cdot L_-(p_i)\exp(\gamma-\gamma_0)H}{(2\pi)^{1/2}(p_i-\alpha_0)\gamma}$$

$$\left[(\alpha+k)^{1/2}\cdot\frac{L_+(\alpha)}{\alpha+\alpha_0}+\frac{(k-\alpha)^{1/2}\ L_+(-\alpha)}{\alpha-\alpha_0}\right]$$

$$-\frac{iH\,A_0(\alpha_0+k)L_-(-\alpha_0)\exp(\gamma-\gamma_0)H}{(2\pi)^{1/2}\gamma}\left[\frac{(\alpha+k)L_+(\alpha)}{\alpha+\alpha_0}+\frac{(k-\alpha)L_+(-\alpha)}{\alpha-\alpha_0}\right]$$

$$+\frac{i\gamma_0 A_0\exp(\gamma-\gamma_0)H}{(2\pi)^{1/2}\gamma(\alpha_0+k)L_+(\alpha_0)}\left[\frac{(\alpha+k)L_+(\alpha)}{\alpha-\alpha_0}+\frac{(k-\alpha)L_+(-\alpha)}{\alpha+\alpha_0}\right]. \tag{77}$$

By the inverse Fourier transforms of (12), we have

$$\phi(x,z)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty+i\tau}^{\infty+i\tau}\overline{\phi}(\alpha,z)\exp(-i\alpha x)\,dx$$

$$=\frac{1}{(2\pi)^{1/2}}\int_{-\infty+i\tau}^{\infty+i\tau}A(\alpha)\exp(-\gamma z-i\alpha x) \tag{78}$$

where

$\tau=\mathrm{Im}(\alpha_0)$, $\alpha\neq\alpha_0$ and $A(\alpha)$ is given by (77).

5. Evaluation of the integral

To evaluate integral (78) the contour is taken along the line $\tau = \mathrm{Im}(\alpha_0)$ as shown in figure 2 avoiding the point $\alpha = \alpha_0$. $\alpha = -\alpha_0$ is the pole and $\alpha = \pm k$ are the branch points and $\mathrm{Re}(\gamma) \geqslant 0$ for all α. The condition $\mathrm{Re}(\gamma) = 0$, as discussed by Ewing and Press [3] gives the parts of hyperbola to be used as branch cut with branch points at $\alpha = \pm k$. Applying Cauchy's residue theorem to the contour chosen, we get

$$\int_{-\infty+i\tau}^{\infty+i\tau} + \int_{AB} + \int_{\Sigma} + \int_{L_\alpha} + \int_{CD} = 2\pi i \text{ (sum of residues)}. \tag{79}$$

The presence of factor $\exp(-i\alpha x) = \exp(-i\sigma x)\exp(-\tau x)(\alpha = \sigma - i\tau)$, makes the integral along infinite circular arcs AB and CD vanish. Σ is a small circle indenting the point $\alpha = \alpha_0$ and L_α is the branch cut around $\alpha = -k$. The contribution of the pole at $\alpha = -\alpha_0$ is

$$-A_0 \exp(-\gamma_0 z)\cdot\exp(i\alpha_0 x), \quad (p_i = -k). \tag{80}$$

This is the reflected wave in the region $x > 0$. From (79), we obtain

$$\frac{1}{(2\pi)^{1/2}}\int_{-\infty+i\tau}^{\infty+i\tau} A(\alpha)\exp(-\gamma z)\cdot\exp(-i\alpha x)\,d\alpha$$

$$= -A_0\exp(-\gamma_0 z)\cdot\exp(i\alpha_0 x) - \frac{1}{(2\pi)^{1/2}}\int_{L_\alpha} A(\alpha)\exp(-\gamma z)\cdot\exp(-i\alpha x)\,d\alpha$$

$$-\frac{1}{(2\pi)^{1/2}}\int_{\Sigma} A(\alpha)\exp(-\gamma z)\cdot\exp(-i\alpha x)\,d\alpha. \tag{81}$$

Indentation around $\alpha = \alpha_0$ (Roos [6]) contributes

$$\phi_1(x, z) = \frac{A_0}{2}\exp(-i\alpha_0 x)\cdot\exp(-\gamma_0 z), \qquad x > 0, \tag{82}$$

$$= -\frac{A_0}{2}\exp(-i\alpha_0 x)\cdot\exp(-\gamma_0 z), \qquad x < 0. \tag{83}$$

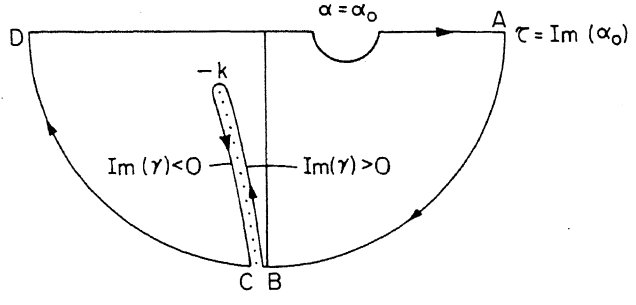

Figure 2.

The term in (82) is the incident wave and in (83) is the wave transmitted to the side $x < 0$.

We now consider

$$I(x, z) = \frac{1}{(2\pi)^{1/2}} \int_{L_\alpha} A(\alpha) \exp(-\gamma z) \cdot \exp(-i\alpha x)\, d\alpha. \tag{84}$$

On $L_\alpha \mathrm{Re}(\gamma) = 0$, and $\mathrm{Im}(\gamma)$ has different values on the two sides of the branch cut. Integrating along two sides of the branch cut, we get

$$I(x, z) = -\frac{1}{\pi} \int_{-k}^{-i\infty} \left\{ \left[\frac{L_-(p_i) 2i\alpha_0 \gamma_0 A_0 H (p_i - k)^{1/2} \ (k-\alpha)^{1/2}}{(p_i - \alpha_0)(\alpha - \alpha_0)} \right.\right.$$

$$\left. - \frac{iHA_0(\alpha_0 + k)(k-\alpha) L_-(-\alpha_0)}{(\alpha - \alpha_0)} + \frac{i\gamma_0 A_0 (k-\alpha)}{(\alpha_0 + k) L_+(\alpha_0)(\alpha + \alpha_0)} \right]$$

$$\cosh \gamma_1 (z - H) L_+(-\alpha) + \left[\frac{2i\alpha_0 \gamma_0 A_0 (p_i - k)^{1/2} \ (\alpha + k)^{1/2} \ L_-(p_i)}{(p_i - \alpha_0)(\alpha + \alpha_0)} \cosh \gamma_1 z \right.$$

$$\left. + \left(\frac{iA_0(\alpha_0 + k)(\alpha + k) L_-(-\alpha_0)}{\alpha + \alpha_0} - \frac{i_0 A_0 (\alpha + k)}{H(\alpha_0 + k)(\alpha - \alpha_0) L_+(\alpha_0)} \right) \sinh \gamma_1 z \right]$$

$$\left. \times \frac{\cosh \gamma_1 H}{L_-(\alpha)\gamma_1} \right\} \frac{\exp(-\gamma_0 z) \exp(-i\alpha x)}{\gamma_1} d\alpha. \tag{85}$$

In obtaining (85), we have kept in mind that $L_-(\alpha)$ being analytic in the lower half plane, does not change its value on two sides of the branch cut and it is the factor $(\alpha + k)^{1/2}$ of $\gamma = \pm (\alpha + k)^{1/2} \cdot (\alpha - k)^{1/2}$, which changes sign alongsides of branch cut. We have also used $\gamma = \mathrm{Im}\, \gamma = \gamma_1 > 0$ in the integral (85). The main contribution to the integral comes from the neighbourhood of the branch point $\alpha = -k$. To find it, we put $\alpha = -k - iu$, so that the neighbourhood of $\alpha = -k$, transforms into neighbourhood of $u = 0$. Under the substitution $\alpha = -k - iu$, the branch integral (85) transforms into a line integral as

$$I(x, z) = \frac{\exp(ikx - \gamma_0 H)}{-\pi} \int_0^\infty \left\{ \left[\frac{-2\alpha_0 \gamma_0 A_0 H (p_i - k)^{1/2} (2k + iu)^{1/2} L_-(p_i)}{(p_i - \alpha_0)(k + iu + \alpha_0)} \right.\right.$$

$$\left. + \frac{HA_0(\alpha_0 + k)(2k + iu) L_-(-\alpha_0)}{k + iu + \alpha_0} - \frac{\gamma_0 A_0 (2k + iu)}{(\alpha_0 + k)(k + iu - \alpha_0) L_+(\alpha_0)} \right]$$

$$\times \cosh \gamma_1 (z - H) L_+(k + iu)$$

$$+ \left[\frac{-2\alpha_0 \gamma_0 A_0 (p_i - k)^{1/2} (-iu)^{1/2} L_-(p_i) \cosh \gamma_1 z}{(k + iu - \alpha_0)(p_i - \alpha_0)} \right.$$

$$\left. + \left(\frac{A_0(\alpha_0 + k)(iu) L_-(-\alpha_0)}{k + iu - \alpha_0} - \frac{\gamma_0 A_0 (iu)}{H(\alpha_0 + k)(k + iu + \alpha_0) L_+(\alpha_0)} \right) \sinh \gamma_1 z \right]$$

$$\left. \times \frac{\cosh \gamma_1 H}{L_-(-k - iu)\gamma_1} \right\} \frac{\exp(-ux)}{\gamma_1} du. \tag{86}$$

This can be written as

$$I(x,z) = -\frac{\exp(ikx-\gamma_0 H)}{\pi}\int_0^\infty \left\{\left[\frac{G_1(u)}{\gamma_1}\cosh\gamma_1(z-H)\right]\right.$$
$$+\frac{G_2(u)}{\gamma_1}[\cosh\gamma_1(z+H)+\cosh\gamma_1(z-H)]$$
$$\left.+G_3(u)[\sinh\gamma_1(z+H)+\sinh\gamma_1(z-H)]\right\}\exp(-ux)\,du, \tag{87}$$

where

$$G_1(u) = \left[\frac{-2\alpha_0\gamma_0 A_0 H(p_i-k)^{1/2}\cdot(2k+iu)^{1/2}\cdot L_-(p_i)}{(p_i-\alpha_0)(k+iu+\alpha_0)}\right.$$
$$+\frac{HA_0(\alpha_0+k)(2k+iu)L_-(-\alpha_0)}{(k+iu+\alpha_0)}$$
$$\left.-\frac{\gamma_0 A_0(2k+iu)}{(\alpha_0+k)(k+iu-\alpha_0)L_+(\alpha_0)}\right]L_+(k+iu),$$
$$G_2(u) = -\frac{\alpha_0\gamma_0 A_0(p_i-k)^{1/2}\cdot(-iu)^{1/2}\cdot L_-(p_i)}{(p_i-\alpha_0)(k+iu-\alpha_0)L_+(-k-iu)}, \tag{88}$$
$$G_3(u) = \frac{i}{2}\left[\frac{A_0(\alpha_0+k)L_-(-\alpha_0)}{(k+iu-\alpha_0)}-\frac{\gamma_0 A_0}{H(\alpha_0+k)(k+iu+\alpha_0)L_+(\alpha_0)}\right]$$
$$\times\frac{u}{\gamma_1^2 L_-(-k-iu)}.$$

Further

$$\gamma^2 = 2iku-u^2 = 2ik_1u-2k_2u-u^2. \tag{89}$$

Since the real part of the γ is zero; therefore from (89), we have

$$k_1 = 0, \quad \gamma^2 = -2k_2u, \text{ for all small } u. \tag{90}$$

To evaluate the integral (87) in the neighbourhood of $u = 0$, we expand $G_1(u)$, $G_2(u)$ and $G_3(u)$ around $u = 0$, i.e.

$$G_1(u) = G_1(0)+uG_1'(0)+\frac{u^2}{\underline{|2}}G''(0)+\ldots \tag{91}$$

and similar expressions of $G_2(u)$ and $G_3(u)$. Since u is small, we retain $G_1(0)$, $G_2(0)$ and $G_3(0)$ and neglect the rest in expansion of $G_1(u)$, $G_2(u)$ and $G_3(u)$. The integral (87) results in

$$I(x,z) \sim -\frac{\exp(ikx-\gamma_0 H)}{i\pi}\int_0^\infty\left\{\frac{G_1(0)}{(2k_2u)^{1/2}}\cos(2k_2u)^{1/2}\ (z-H)\right.$$
$$\left.+\frac{G_2(0)}{(2k_2u)^{1/2}}\cdot[\cos(2k_2u)^{1/2}(z+H)+\cos(2k_2u)^{1/2}(z-H)]\right.$$

$$- G_3(0)\,[\sin\,(2k_2u)^{1/2}(z+H)+\sin\,(2k_2u)^{1/2}(z-H)]\Big\}\exp(-ux)\,\mathrm{d}u. \tag{92}$$

To evaluate the integrals in (92), we use the results (Sneddon, [7] p. 200)

$$\int_0^\infty \sin\,(u)^{1/2}x\exp(-ktu)\,\mathrm{d}u = \frac{(\pi)^{1/2}x\exp(-x^2/4kt)}{2(kt)^{3/2}}, \tag{93}$$

$$\int_0^\infty \frac{\cos\,(u)^{1/2}\,x}{(u)^{1/2}}\exp(-ktu)\,\mathrm{d}u = \frac{(\pi)^{1/2}}{(kt)^{1/2}}\exp(-x^2/4kt), \tag{94}$$

We obtain

$$\begin{aligned} I(x,z) \sim -\frac{\exp(ikx-\gamma_0H-i\pi/2)}{(2k_2\pi x)^{1/2}}\Bigg[&[G_1(0)\exp(-k_2(z-H)^2/2x)] \\ &+G_2(0)[\exp(-k_2(z+H)^2/2x)+\exp[-k_2(z-H)^2/2x]] \\ &-G_3(0)\Bigg[\frac{k_2(z+H)}{x}\exp(-k_2(z+H)^2/2x) \\ &+\frac{k_2(z-H)}{x}\exp(-k_2(z-H)^2/2x)\Bigg]\Bigg] \end{aligned} \tag{95}$$

At points at large distances from the barrier, we have

$$-k_2[x^2+(z\pm H)^2]^{1/2} \sim -k_2x-k_2(z\pm H)^2/2x.$$

Then

$$\begin{aligned} I(x,z) = \frac{\exp(-\gamma_0H-i\pi/2)}{(2k_2\pi x)^{1/2}}\Big\{&G_1(0)\exp(-k_2[x^2+(z-H)^2]^{1/2}) \\ &+G_2(0)[\exp(-k_2[x^2+(z+H)^2]^{1/2})+\exp(-k_2[x^2 \\ &+(z-H)^2]^{1/2})-\frac{k_2}{x}G_3(0)[(z+H)\exp(-k_2[x^2+(z+H)^2]^{1/2}) \\ &+(z-H)\exp(-k_2[x^2+(z-H)^2]^{1/2})\Big\} \end{aligned} \tag{96}$$

$$\begin{aligned} = \frac{\exp(-\gamma_0H-i\pi/2)}{(2k_2\pi x)^{1/2}}\Big\{&G_1(0)\exp(-k_2r_1)+G_2(0)[\exp(-k_2r_2) \\ &+\exp(-k_2r_1)]-k_2G_3(0)\Bigg[\frac{(z+H)}{x}\exp(-k_2r_2) \\ &+\frac{(z-H)}{x}\exp(-k_2r_1)\Bigg]\Big\}, \end{aligned} \tag{97}$$

where

$$r_1^2 = x^2+(z-H)^2, \tag{98}$$

$$r_2^2 = x^2+(z+H)^2, \tag{99}$$

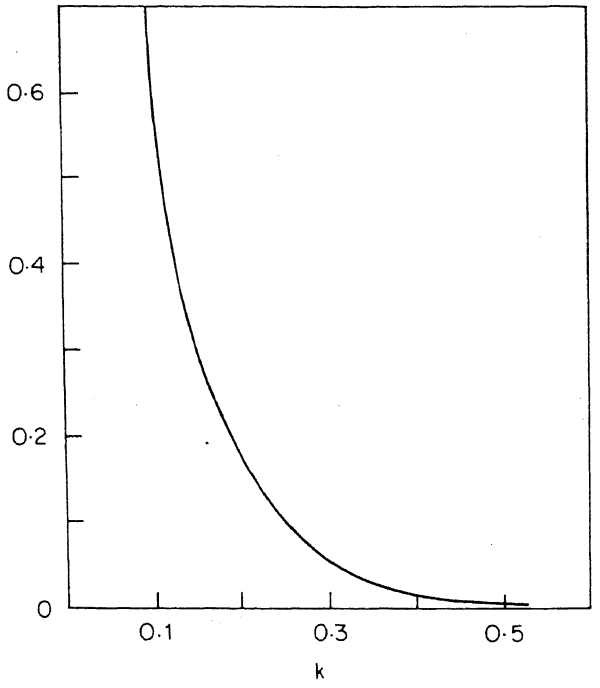

Figure 3.

and

$$G_1(0) = \left[-\frac{2\alpha_0\gamma_0 A_0 H(p_i-k)^{1/2}(2k)^{1/2} L_-(p_i)}{(p_i-\alpha_0)(k+\alpha_0)} \right.$$

$$\left. + \frac{HA_0(\alpha_0+k)(2k)L_-(-\alpha_0)}{k+\alpha_0} - \frac{\gamma_0 A_0 2k}{(\alpha_0+k)(k-\alpha_0)L_+(\alpha_0)} \right] L_+(k), \tag{100}$$

$$G_2(0) = \frac{\alpha_0\gamma_0 A_0(p_i-k)^{1/2}(-i)^{1/2} L_-(p_i)}{(k-\alpha_0)(p_i-\alpha_0)i(2k)^{1/2} L_-(-k)}, \tag{101}$$

$$G_3(0) = \frac{i}{2}\left[\frac{A_0(\alpha_0+k)L_-(-\alpha_0)}{(k-\alpha_0)} - \frac{\gamma_0 A_0}{H(\alpha_0+k)(k+\alpha_0)L_+(\alpha_0)} \right]$$

$$\times \frac{1}{(-2k)L_-(-k)}. \tag{102}$$

6. Results

We now summarize the results obtained in the present paper. The reflected wave in region $x > 0$ is given by (80). Its magnitude is the same as that of the incident wave. The wave transmitted to the region $x < 0$ is given by (83) and has half the amplitude of

the incident wave. The scattered waves are obtained in (97). They are of the form $\exp(-k_2 r_1)/(r_1)^{1/2}$ or $\exp(-k_2 r_2)/(r_2)^{1/2}$ where $r_1 = [x^2 + (z-H)^2]^{1/2}$ and $r_2 = [x^2 + (z+H)^2]^{1/2}$. These are cylindrical waves originating at the edge $(0, H)$ of the barrier and at its image $(0, -H)$ in the free surface. The graph of the amplitude of the scattered wave in terms of the wave number k is given by figure 3. It has a large value for $k = 0$ and decreases rapidly as k increases. The results are obtained when $H = 0.1$ km at a point $r = 10$ km in the free surface of the oceanic liquid.

References

[1] Deshwal P S Diffraction of a compressional wave by a rigid barrier in a liquid half-space, Pure and Applied Geophysics, 85 (1971/II), 107–124.
[2] Deshwal P S Diffraction of a compressional wave by a rigid barrier in a liquid layer—part II, Pure and Applied Geophysics, 91 (1971/VIII), 14–33.
[3] Ewing W Jardetzky W S and Press F Elastic waves in layered media, (1957), McGraw Hill Book Co.
[4] Faulkner T R Proc. Camb. Phil. Soc., 62 (1966), 829–838.
[5] Noble B Methods based on the Wiener-Hopf technique, (1958), Pergamon Press.
[6] Roos B W Analytic functions and distributions in Physics and Engineering, (1969), John Wiley and Sons, Inc. New York.
[7] Sneddon I N Fourier Transforms, (1951), McGraw Hill Book Co.
[8] Ursell F Proc. Camb. Phil. Soc., 43 (1947), 374—382.

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Schiffer variation of complex structure and coordinates for Teichmüller spaces

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Abstract. Schiffer variation of complex structure on a Riemann surface X_0 is achieved by punching out a parametric disc $\overline{D}$ from X_0 and replacing it by another Jordan domain whose boundary curve is a holomorphic image of $\partial\overline{D}$. This change of structure depends on a complex parameter ε which determines the holomorphic mapping function around $\partial\overline{D}$.

It is very natural to look for conditions under which these ε-parameters provide local coordinates for Teichmüller space $T(X_0)$, (or reduced Teichmüller space $T^{\#}(X_0)$). For compact X_0 this problem was first solved by Patt [8] using a complicated analysis of periods and Ahlfors' [2] τ-coordinates.

Using Gardiner's [6], [7] technique, (independently discovered by the present author), of interpreting Schiffer variation as a quasi conformal deformation of structure, we greatly simplify and generalize Patt's result. Theorems 1 and 2 below take care of all the finite-dimensional Teichmüller spaces. In Theorem 3 we are able to analyse the situation for infinite dimensional $T(X_0)$ also. Variational formulae for the dependence of classical moduli parameters on the ε's follow painlessly.

Keywords. Riemann surfaces; Teichmüller spaces; quasiconformal mappings.

1. Introduction

We are interested in making explicit variations of complex-structure on a Riemann surface X_0 so that the variation parameters provide complex-analytic and real-analytic coordinates (respectively) on the Teichmüller space $T(X_0)$ and reduced Teichmüller space $T^{\#}(X_0)$. Such variations, obtained by changing the complex structure on disjoint discs in X_0, were introduced by Schiffer, see [9].

In two interesting papers Gardiner [6], [7], showed that Schiffer's variation can be achieved by quasiconformal (q.c.) deformation, and that Schiffer's variational formulae are equivalent to q.c. variational formulae involving appropriate Beltrami differentials. The technique is applied in the present article to give a very general and simple solution to the coordinatisation problem for moduli space mentioned at the beginning.

Instead of using periods and Ahlfors' τ-coordinates as in Patt [8], we use Bers coordinates for our analysis. We prove that if Schiffer variations are carried out independently in d suitably-chosen disjoint discs on X_0, with arbitrarily specified boundaries and/or almost-arbitrarily specified centres, then the ε-parameters provide

local complex-analytic coordinates for $T(X_0)$ around X_0. See Theorem 1. Here d is the (finite) complex dimension of $T(X_0)$.

Even when X_0 is not of finite conformal type, but the reduced space $T^{\#}(X_0)$ is a d-dimensional real-analytic manifold, we can use the real parts of the ε's as local real-analytic coordinates for $T^{\#}(X_0)$, (Theorem 2).

In §5 we have a theorem for infinite dimensional Teichmüller spaces using a countable family of discs for variation of structure on X_0. That such an analysis is possible testifies again to the power of interpreting Schiffer variation as q.c. deformation.

2. Preliminaries

Let X_0 be an arbitrary Riemann surface and t a (holomorphic) local parameter around a point $p \in X_0$. Without loss of generality we assume that $t(p) = 0$ and that the image of t contains a disc of radius greater than one around 0. We call the open domain $D = t^{-1}(\Delta)$ a parametric unit disc on X_0 with centre p, (where Δ is the open unit disc in $\mathbf{C}$).

We denote the boundary of D by $\partial D = \beta = \{x \in X_0 : |t(x)| = 1\}$. Note that, owing to the profusion of conformal Riemann mappings, the Jordan curve β on X_0 can be chosen with a great degree of arbitrariness.

A new Riemann surface, X_ε^*, will be defined by making the following 'Schiffer variation' of complex structure on the disc D. Indeed,

$$t^*(t) = t + \frac{\varepsilon}{t}, \quad \varepsilon \in \mathbf{C} \tag{1}$$

is a holomorphic function in an annular neighbourhood of β and maps β to a Jordan curve β^* in the t^*-plane for small ε. The Jordan domain with boundary β^* is denoted D^*; D^* is of course a bounded simply-connected region of the t^*-plane.

X_ε^* is obtained now by removing D from X_0 and filling in the hole with $\overline{D}^*$ (bar denotes closure)—the boundary identification being given by (1). So x on β is identified with $t^*(t(x))$ on β^*. On $\overline{D}^*$ we use t^* as a holomorphic coordinate, and on $X_\varepsilon^* - \overline{D}^* = X_0 - \overline{D}$ we use the original coordinates from X_0. Note that on $\partial \overline{D}^* = \beta^* \subset X_\varepsilon^*$ we may use either t or t^* as holomorphic coordinates. Clearly X_ε^* becomes a well-defined Riemann surface topologically equivalent (but in general *not* conformally equivalent) to X_0. Obviously, if X_0 is topologically marked (by a choice of generators for $\pi_1(X_0)$) so is X_ε^*.

From now on let $X_0 = U/G$, G a torsion-free Fuchsian group operating on the upper half-lane, U, or on the unit disc Δ, (whichever is convenient). We recall briefly relevant points regarding the Teichmüller space $T(X_0) = T(G)$ and reduced Teichmüller space $T^{\#}(X_0) = T^{\#}(G)$.

For this purpose let κ denote the holomorphic cotangent bundle of X_0. A Beltrami differential μ on X_0 is a L^∞ section of the bundle $\bar{\kappa} \otimes \kappa^{-1}$ over X_0, so it is represented in local parameters on X_0 by

$$\mu = \mu(z)\frac{d\bar{z}}{dz}, \ \|\mu\|_\infty < \infty.$$

We call the complex Banach space of Beltrami differentials $L^\infty(X_0) = L^\infty(G) = L^\infty(\bar\kappa \otimes \kappa^{-1})$. The open unit ball in $L^\infty(X_0)$ is denoted $M(X_0) = M(G)$ and is called the Banach manifold of proper Beltrami differentials.

Any $\mu \in M(X_0)$ defines a 'Riemannian metric' $\lambda|dz + \mu d\bar z|$, whose conformal class gives X_0 a conformal (= complex) structure. Indeed, local homeomorphic solutions of the Beltrami equation $\bar\partial w = \mu \cdot \partial w$, with the coefficient μ, provide holomorphic local coordinates for the new complex structure. X_0 with this complex structure is denoted X_μ.

Now, if $\varphi : X_0 \to Y$ is a q.c. homeomorphism onto another Riemann surface Y, then the complex dilatation of φ, denoted $(\mu(\varphi)(= \bar\partial\varphi/\partial\varphi)$, forms a proper Beltrami differential on X_0. Indeed φ becomes biholomorphic from $X_{\mu(\varphi)}$ to Y.

We define $\mu, \nu \in M(X_0)$ to be equivalent ($\sim$) if there is a biholomorphism between X_μ and X_ν homotopic to the identity where throughout the homotopy the ideal boundary of X_0 remains pointwise fixed. We define μ and ν to be weakly equivalent ($\tilde{\#}$) if the condition for this homotopy on the ideal boundary is dropped. We set

$$T(X_0) = M(X_0)/\sim \text{ and } T^\#(X_0) = M(X_0)/\tilde{\#}.$$

Both spaces parametrize marked Riemann surfaces which are quasiconformally homeomorphic to X_0. We denote the natural projections from $M(X_0)$ to $T(X_0)$ and $T^\#(X_0)$ by Φ and $\Phi^\#$ respectively. $T(X_0)$ itself of course projects onto the (usually smaller) space $T^\#(X_0)$.

If X_0 is of finite type (g, k), (i.e. a compact genus g surface with k deleted (or distinguished) points), then $T(X_0) \equiv T^\#(X_0)$ inherits a (unique) complex structure of a $(3g-3+k)$-dimensional complex mainfold making Φ a holomorphic submersion. If X_0 is not of finite type but $G = \pi_1(X_0)$ is finitely generated, then the Schottky double $\hat X_0$ of X_0 is of finite type (g', k'), and $T^\#(X_0)$ embeds ('by doubling') as a real analytic manifold of real dimension $(3g'-3+k')$ in $T(\hat X_0)$—the latter being a complex manifold of the same number of complex dimensions. These are the only situations where $T(X_0)$ or $T^\#(X_0)$ are finite-dimensional (Earle [4]).

Let $Q(X_0)$ denote the integrable holomorphic quadratic differentials on X_0, i.e. the holomorphic sections ψ of $\kappa \otimes \kappa$ over X_0 such that the L^1-norm is finite:

$$\|\psi\| = \iint_{X_0} |\psi(z)| dx dy < \infty. \tag{2}$$

Of course $Q(X_0) \subset L^1(\kappa \otimes \kappa)$, and this latter Banach space has the usual duality-pairing with $L^\infty(\bar\kappa \otimes \kappa^{-1}) = L^\infty(X_0)$ by

$$\langle \psi, \mu \rangle = \iint_{X_0} \psi \mu dz \hat A d\bar z, \quad \psi \in L^1(\kappa \otimes \kappa),\ \mu \in L^\infty(\bar\kappa \otimes \kappa^{-1}). \tag{3}$$

In case X_0 is of type (g, k), $Q(X_0)$ is a complex vector space of dimension equal to the complex dimension of $T(X_0)$, (Riemann-Roch). In any case $T(X_0)$ is well known to be a complex Banach manifold and $T^\#(X_0)$ a real Banach manifold with Φ and $\Phi^\#$ analytic submersions. We require the following classical 'Teichmüller's Lemma' and a variant:

LEMMA 1. The kernel of the differential of Φ at $\mu = 0$ is

$$N(X_0) = Q(X_0)^{\perp} = \{v \in L^{\infty}(X_0): \langle \psi, v \rangle = 0, \text{ for all } \psi \in Q(X_0)\}.$$

Thus the holomorphic tangent space to $T(X_0)$ at X_0 is $Q(X_0)^* = L^{\infty}(X_0)/N(X_0)$.

The embedding of $T^{\#}(X_0)$ in $T(\hat{X}_0)$ is by extending $\mu \in M(X_0)$ to $\mu^{\text{ext}} \in M(\hat{X}_0)$ using the obvious reflection.

LEMMA 2. The kernel of the differential of $\Phi^{\#}$ at $\mu = 0$ is

$$N^{\#}(X_0) = \{v \in L^{\infty}(X_0): \langle \psi, v^{\text{ext}} \rangle = 0, \quad \text{for all } \psi \in Q(\hat{X}_0)\}.$$

The real-analytic tangent space to $T^{\#}(X_0)$ at X_0 is $L^{\infty}(X_0)/N^{\#}(X_0)$. $Q(\hat{X}_0)$ is the (real) direct sum of two copies of $Q^{\#}(X_0)$, where $Q^{\#}(X_0)$ comprises those integrable holomorphic quadratic differentials on X_0 which are real on the ideal boundary of X_0. Clearly, $L^{\infty}(X_0)/N^{\#}(X_0)$ is the real dual space of $Q^{\#}(X_0)$. In fact, μ in $L^{\infty}(X_0)$ acts on $Q^{\#}(X_0)$ as the linear functional

$$l_{\mu}(\psi) = \text{Re}\left\{\iint_{X_0} \psi \mu \, dx dy\right\}.$$

For Lemma 1 see Ahlfors [1], and for Lemma 2 see Earle [4, p. 60].

Suppose $T(X_0)$ is finite dimensional and $\{\mu_1, \ldots, \mu_d\}$ is a $\mathbf{C}$-basis for $L^{\infty}(X_0)/N(X_0)$. Then clearly, by Lemma 1, the map from a neighbourhood of the origin in $\mathbf{C}^d$ to $T(X_0)$ which sends

$$(\tau_1, \ldots, \tau_d) \mapsto \Phi(\tau_1 \mu_1 + \ldots + \tau_d \mu_d) \tag{4}$$

is (the inverse of) a holomorphic coordinate system for a neighbourhood of X_0 in $T(X_0)$. The $(\tau_1, \ldots, \tau_d)$ are called 'Bers coordinates'. An analogous statement holds for real-analytic coordinates in a finite-dimensional $T^{\#}(X_0)$ using Lemma 2.

3. Two main theorems

On a marked Riemann surface X_0 we carry out independent Schiffer variations in $n(\geqslant 1)$ disjoint parametric unit discs $D_1, \ldots, D_n$ centred at $p_1, \ldots, p_n$ with parameter t_k in D_k. $\partial D_k = \beta_k$ is mapped to β_k^* as in (1) by:

$$t_k^*(t_k) = t_k + \frac{\varepsilon_k}{t_k}.$$

The new marked Riemann surface is

$$X_{\varepsilon}^* = X_{\varepsilon_1, \ldots, \varepsilon_n}^* \text{ in } T(X_0).$$

The double $\hat{X}_{\varepsilon}^*$ of X_{ε}^* is an element of $T^{\#}(X_0) \subsetneq T(\hat{X}_0)$. Here ε denotes $(\varepsilon_1, \ldots, \varepsilon_n)$.

THEOREM 1. For Schiffer variation on n disjoint discs as above the map S:

$$(\varepsilon_1, \ldots, \varepsilon_n) \overset{S}{\mapsto} X_\varepsilon^* \tag{5}$$

is holomorphic from a neighbourhood of 0 in $\mathbf{C}^n$ into $T(X_0)$.

If $d =$ complex dimension of $T(X_0)$ is finite, then, given any d points $\{p_1, \ldots, p_d\}$ on X_0 it is possible to choose parametric unit discs with centres $\{p_1', \ldots, p_d'\}$ lying in arbitrarily small neighbourhoods of the original points so that the variation parameters $(\varepsilon_1, \ldots, \varepsilon_d)$ are holomorphic coordinates for $T(X_0)$ around X_0.

Indeed, if we specify d disjoint parametric unit discs on X_0 with boundaries $\{\beta_1, \ldots, \beta_d\}$, it is possible to choose local parameters for these very discs so that the corresponding ε's again provide holomorphic local coordinates on $T(X_0)$.

The variation parameters corresponding to parametric discs centred at any $\{p_1, \ldots, p_d\}$ are local coordinates if and only if any ψ in $Q(X_0)$ that vanishes at each p_k vanishes identically.

Remark. It is noteworthy that the last statement, which is a corollary of the proof, depends only on the points p_k and *not* on the local parameters.

THEOREM 2. For Schiffer variations in any n disjoint discs on X_0 the map $S^\#$:

$$(\varepsilon_1, \ldots, \varepsilon_n) \overset{S^\#}{\mapsto} \hat{X}_\varepsilon^*, \ (\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)), \tag{6}$$

from a neighbourhood of 0 in $\mathbf{C}^n$ into $T^\#(X_0)\,(\subsetneq T(\hat{X}_0))$ is real-analytic.

If $d =$ real dimension of $T^\#(X_0)$ is finite, then it is possible to choose d disjoint parametric unit discs on X_0 so that the real parts of $(\varepsilon_1, \ldots, \varepsilon_d)$ provide real-analytic local coordinates for a neighbourhood of X_0 in $T^\#(X_0)$.

Once again the centres of the discs can be required to lie in arbitrarily small open regions, and/or the boundaries $\{\beta_1, \ldots, \beta_d\}$ of the variation-discs can be prescribed beforehand.

The real-parts of the variation parameters for the t_k-discs D_k, centered at p_k, $k = 1, \ldots, d$, provide local coordinates if and only if any ψ in $Q^\#(X_0)$ whose local expressions $\psi_k(t_k)\,dt_k^2$ satisfy $\mathrm{Re}(\psi_k(0)) = 0$, (each $k = 1, \ldots, d$) identically vanishes. This time the condition depends not only on the centres of the discs but also on the local parameters.

Proof of Theorem 1

We only need to show S holomorphic with respect to each ε_j separately, so we may restrict attention to variation in one disc D with parameter t. As in Gardiner [6], we produce an explicit q.c. homeomorphism $\varphi_\varepsilon : X_0 \to X_\varepsilon^*$. In fact, let

$$t^* = \varphi_\varepsilon(t) = t + \varepsilon\bar{t} \text{ on } |t| \leqslant 1.$$

It is easy to check that φ_ε maps $\overline{D}$ onto $\overline{D}^*$ with the correct boundary identification, and φ_ε is a C^∞ diffeomorphism for $|\varepsilon| < 1$. (Note, φ_ε maps the radius vector to exp $(i\theta)$

proportionally upon the radius vector to $t^*[\exp(i\theta)]$. Thus:

$$\varphi_\varepsilon = \begin{cases} t + \varepsilon\bar{t} \text{ on } \overline{D} \\ \text{Identity on } X_0 - D \end{cases} \tag{7}$$

is clearly a marking-preserving q.c. homeomorphism of X_0 onto X_ε^*.
The complex dilatation of φ_ε is $\mu(\varphi_\varepsilon) \in M(X_0)$ where

$$\mu(\varphi_\varepsilon) = \begin{cases} \varepsilon\dfrac{d\bar{t}}{dt} & \text{on } D, \\ 0 \text{ on } X_0 - D, & \|\mu(\varphi_\varepsilon)\|_\infty = |\varepsilon| < 1. \end{cases} \tag{8}$$

Since $\mu(\varphi_\varepsilon)$ evidently depends holomorphically on ε and

$$S(\varepsilon) = X_{\mu(\varphi_\varepsilon)} = \Phi(\mu(\varphi_\varepsilon))$$

we see that S is holomorphic.

Suppose now that independent variations are carried out in n disjoint parametric discs $D_1, \ldots, D_n$. We see then:

$$X^*_{\varepsilon_1, \ldots, \varepsilon_n} = \Phi(\varepsilon_1\mu_1 + \ldots + \varepsilon_n\mu_n) \tag{9}$$

where,

$$\mu_k = \begin{cases} d\bar{t}_k/dt_k \text{ on } D_k, \\ 0 \text{ on } X_0 - D_k \end{cases} \quad k = 1, \ldots, n. \tag{10}$$

Therefore, by definition of the Bers coordinates, $(\varepsilon_1, \ldots, \varepsilon_d)$ will be holomorphic coordinates for $T(X_0)$ precisely when the $\{\mu_1, \ldots, \mu_d\}$ given by (10) form a $\mathbf{C}$-basis for $L^\infty(X_0)/N(X_0) = Q(X_0)^*$.

The special form of our Beltrami differentials in (10) shows that μ_k, as an element of $Q(X_0)^*$, is the linear functional

$$l_k(\psi) = -2i\pi\psi_k(0), \tag{11}$$

where $\psi = \psi_k(t_k)dt_k^2$ in the parametric t_k-disc D_k. This is simply because, by the mean value theorem,

$$\langle \psi, \mu_k \rangle = \iint\limits_{|t_k| \leqslant 1} \psi_k(t_k)\, dt_k \wedge d\bar{t}_k = -2i\pi\psi_k(0).$$

Suppose we make a change of parameter for D_k from t_k to $\tilde{t}_k$, $\tilde{t}_k$ being centred at a new point q_k within D_k. Of course, the t_k to $\tilde{t}_k$ transformation is a Möbius automorphism of the unit disc that throws 0 to $t_k(q_k)$. The linear functional $\tilde{l}_k$ in $Q(X_0)^*$, corresponding to Schiffer variation with centre q_k and $\tilde{t}_k$-disc D_k, is of course

$$\tilde{l}_k(\psi) = -2i\pi\tilde{\psi}_k(0).$$

Here $\psi = \tilde{\psi}_k(\tilde{t}_k)d\tilde{t}_k^2$ in the $\tilde{t}_k$ local coordinate. But then the equality $\psi_k(t_k)dt_k^2 = \tilde{\psi}_k(\tilde{t}_k)d\tilde{t}_k^2$ shows that

$$\tilde{l}_k(\psi) = a\psi_k(q_k), \tag{12}$$

upto some non-zero constant a. Since non-zero multiples do not affect linear independence conditions, it is enough to find q_k in the given neighbourhoods of p_k such that the corresponding evaluations at q_k are d **C**-linearly independent functionals on $Q(X_0)$.

This is easy to do as follows.

Claim. For any t_k-disc D_k, and any neighbourhood A_k of the centre of D_k, the linear functionals $l_a(\psi) = \psi_k(a)$, $a \in A_k$, $\psi \in Q(X_0)$, span $Q(X_0)^*$.

Proof. If $\psi_k \equiv 0$ on A_k then ψ itself is identically zero. Now set

$$S_k = \{l_a : a \in A_k\},\ k = 1, \ldots, d.$$

These are subsets of $Q(X_0)^*$ such that each one spans all of $Q(X_0)^*$. All we have to do is to choose d linearly independent vectors $\{\sigma_1, \ldots, \sigma_d\}$, with $\sigma_k \in S_k$. But this is always possible because of the following:

Fact from linear algebra. Let $S_1, \ldots, S_n$ be subsets of any vector space V such that each S_k spans a subspace of dimension at least n. Then there is a set $\{\sigma_1, \ldots, \sigma_n\}$ of n linearly independent vectors in V, σ_k being from S_k for each $k = 1, \ldots, n$.

Proof. A trivial induction on n.

We have evidently completed the proof of all assertions in Theorem 1. From the proof it is clear that both the restrictions on the positions of the centres and the fixing of the boundaries may be imposed simultaneously.

Remark. Notice that no choice of discs can make the ε's global coordinates for $T(X_0)$. This is because otherwise our formula (9) would give a global holomorphic cross-section for the projection Φ, and Earle [5] has shown that this is impossible if $d > 1$.

However, since $T(X_0)$ is arc-connected we see by a compactness argument that one can pass from any complex structure to any other by a finite series of successively applied Schiffer variations carried out in suitably chosen sets of d discs.

Proof of Theorem 2

This theorem is interesting precisely when X_0 is not of finite type but its fundamental group is finitely generated.

Clearly, the q.c. map $\varphi_\varepsilon : X_0 \to X_\varepsilon^*$ extends by reflection to a q.c. map $\hat{\varphi}_\varepsilon : \hat{X}_0 \to \hat{X}_\varepsilon^*$, and the Beltrami coefficient $\mu(\hat{\varphi}_\varepsilon) \in M(\hat{X}_0)$ is the extension by reflection of $\mu(\varphi_\varepsilon) \in M(X_0)$. Thus,

$$\hat{X}_\varepsilon^* = \Phi^\#(\mu(\varphi_\varepsilon)) \tag{13}$$

and clearly therefore, the Schiffer map $S^\#$: Neighbourhood of 0 in $\mathbf{C}^n \to T^\#(X_0)$, is real-analytic.

To prove that the real parts of $(\varepsilon_1, \ldots, \varepsilon_d)$ give real-analytic coordinates on $T^\#(X_0)$ around X_0 we are again reduced to showing that for suitable choice of discs

$\{D_1, \ldots, D_d\}$ on X_0 the Beltrami differentials μ_k of (10) form a **R**-basis for $L^\infty(X_0)/N^\#(X_0)$.

As in the proof of Theorem 1, using Lemma 2 now instead of Lemma 1, we identify the μ_k as real linear functionals l_k on $Q^\#(X_0)$, where

$$l_k(\psi) = \mathrm{Re}(\pi\psi_k(0)). \tag{14}$$

where ψ in $Q^\#(X_0)$ has the local expression $\psi_k(t_k)\mathrm{d}t_k^2$ in the t_k-disc D_k (with centre p_k). This time a change of local parameter, even preserving the centre, can effect a non-trivial change in the corresponding functional. Indeed, l_k gets replaced by

$$l_k'(\psi) = a\pi\,\mathrm{Re}\,[\exp(i\theta)\psi_k(0)], \text{ some real } \theta, \tag{15}$$

where a is a non-zero real constant. (Again a can be ignored for purposes of **R**-linear independence.) Note that any real θ is achievable by suitable change of parameter.

To prove Theorem 2 it is clearly sufficient to demonstrate the existence of q_k in the given neighbourhoods A_k of p_k, and reals θ_k, such that the linear functionals

$$\tilde{l}_k(\psi) = \mathrm{Re}\,[\exp(i\theta_k)\cdot\psi_k(q_k)],\ k = 1, \ldots, d,$$

form a linearly independent set in $(Q^\#(X_0))^*$.

But, as before, the sets

$$S_k = \{l_{q,\theta} \in (Q^\#(X_0))^* : l_{q,\theta}(\psi) = \mathrm{Re}\,[\exp(i\theta)\psi_k(q)],\ q \in A_k,\ \theta \in \mathbf{R}\} \tag{16}$$

span all of $(Q^\#(X_0))^*$ because $\psi_k \equiv 0$ on A_k again implies $\psi \equiv 0$. So the same 'Fact from linear algebra' used in the previous proof does the needful.

All the assertions are now evident.

A question: Can one choose $[\frac{1}{2}(d+1)]$ discs on X_0 so that using d real *and* imaginary parts of the corresponding complex ε's we get real analytic coordinates for $T^\#(X_0)$?

4. Variational formulae

From our analysis Patt's variational formulae follow painlessly. As usual define the period mappings, $\pi_{ij}: T(X_0) \to \mathbf{C}$, by

$$\pi_{ij}(X_\mu) = \int_{b_j} \omega_i,$$

where $(a_1, \ldots, a_g, b_1, \ldots, b_g)$ is the canonical homology basis on the compact genus $g(\geq 2)$ marked Riemann surface $X_\mu \in T(X_0)$, and $(\omega_1, \ldots, \omega_g)$ is the canonical dual basis of holomorphic 1-forms.

Applying the bilinear relations simply for differentials of the first kind, following Ahlfors [2], we can deduce Rauch's variational formula, ((17) below), for π_{ij}, in the tangent direction μ at $X_0 \in T(X_0)$ for any smooth μ. But then, by Teichmüller's Lemma (Lemma 1), the formula (17) must hold for arbitrary bounded measurable Beltrami

differentials μ. This is because, by the Ahlfors-Weill section formula, any tangent direction has a *very* smooth (in fact real-analytic) Beltrami differential as representative.

$$\pi_{ij}(X_{\varepsilon\mu}) - \pi_{ij}(X_0) = \varepsilon\left[\iint_{X_0} (\omega_i \otimes \omega_j)\mu\right] + O(\varepsilon^2) \tag{17}$$

i.e.

$$d_{X_0}\pi_{ij}(d_0\Phi(\mu)) = \langle \omega_i \otimes \omega_j, \mu \rangle$$

We would like to understand the change in π_{ij} with Schiffer variation of complex structure. Let $\omega_i = \omega_i(t)\mathrm{d}t$ in the t-disc D, then we know $\mu(\varphi_\varepsilon)$ as in (8), so:

$$\begin{aligned}\pi_{ij}(X_\varepsilon^*) - \pi_{ij}(X_0) &= \varepsilon \iint_D \omega_i(t)\omega_j(t)\mathrm{d}t \wedge \mathrm{d}\bar{t} + O(\varepsilon^2) \\ &= -2i\pi\varepsilon\omega_i(0)\omega_j(0) + O(\varepsilon^2).\end{aligned} \tag{18}$$

This last result was deduced in Patt [8], (his equation (29)), as one of his central results; he uses differentials of the third kind and a complicated analysis. See Gardiner [6, p. 379] for a similar proof of a somewhat different variation for π_{ij}.

5. Schiffer variations in infinite-dimensional moduli spaces

Consider now X_0 such that $T(X_0)$ and/or $T^\#(X_0)$ is infinite dimensional. Choose countably many disjoint parametric unit discs $(D_1, D_2, \ldots)$ on X_0 with corresponding Schiffer variations $(\varepsilon_1, \varepsilon_2, \ldots) = \varepsilon$. Clearly, as long as

$$\varepsilon \in l^\infty_\Delta = \text{unit ball in the Banach space } l^\infty \text{ of bounded complex sequences}$$

we get our q.c. map $\varphi_\varepsilon : X_0 \to X_\varepsilon^*$ with $\|\mu(\varphi_\varepsilon)\|_\infty \leqslant \|\varepsilon\|_\infty$. Thus we meaningfully define the Schiffer variation maps

$$\begin{aligned}&S : l^\infty_\Delta \to T(X), \text{ and} \\ &S^\# : l^\infty_\Delta \to T^\#(X_0) \subsetneq T(\hat{X}_0)\end{aligned} \tag{19}$$

just as before, ($S^\#$ by doubling X_ε^*).

Now, $T(X_0)$ is a Banach manifold—an open subset (*via* the Bers embedding) of the complex Banach space $B(G)$, ($X_0 = U/G$, G Fuchsian),

$$\begin{aligned}B(G) = \{&\varphi \in \mathrm{Hol}(U) : \|\varphi\| = 4\|\varphi(z)y^2\|_\infty < \infty, \\ &\text{and } \varphi \text{ induces a quadratic form on } X_0\}\end{aligned} \tag{20}$$

Also, $B(G)$ is known to be the dual of the separable Banach space

$$\begin{aligned}A(G) = Q(X_0) = \{&\psi \in \mathrm{Hol}(U) : \iint_{U/G} |\psi| < \infty, \psi \text{ induces a quadratic} \\ &\text{form on } X_0\},\end{aligned} \tag{21}$$

via the usual Weil-Petersson pairing, namely $(\psi, \varphi) = \iint_{U/G} \psi(z)\overline{\varphi(z)}y^2\,dxdy$.

Now, from our knowledge of $\mu(\varphi_\varepsilon)$ we can actually calculate the derivative at 0 of S:

$$d_0S : l^\infty \to Q(X_0)^* \equiv L^\infty(X_0)/Q(X_0)^\perp.$$

Indeed,

$$d_0S(c_1, c_2, \ldots) = \left(c_1\frac{d\bar{t}_1}{dt_1} + c_2\frac{d\bar{t}_2}{dt_2} + \ldots\right) \bmod Q(X_0)^\perp \tag{22}$$

as is clear since $S(\varepsilon) = \Phi(\mu(\varphi_\varepsilon))$.

Consider the following bounded linear map

$$\theta : Q(X_0) \to l^1 \tag{23}$$

given by integration over the discs D_j:

$$\theta(\psi) = \left(\iint_{D_1}\psi, \iint_{D_2}\psi, \ldots\right) \tag{24}$$

where $\iint_{D_k}\psi$ is of course $\iint_{|t_k|\leqslant 1}\psi_k(t_k)dt_k \wedge d\bar{t}_k = -2i\pi\psi_k(0)$. Obviously, the operator norm $\|\theta\| \leqslant 1$.

THEOREM 3. The map d_0S of (22) is precisely the dual of the map θ of (24). Consequently, the Schiffer variation map S provides local holomorphic coordinates to $T(X_0)$ around X_0 if and only if θ is an isomorphism of Banach spaces.

Proof. Let $c = (c_1c_2, \ldots) \in l^\infty$. Then c determines a Beltrami differential μ_c in $L^\infty(X_0)$ by

$$\mu_c = \begin{cases} c_1\dfrac{d\bar{t}}{dt_1} \text{ on } D_1 \\ \vdots \\ c_k\dfrac{d\bar{t}_k}{dt_k} \text{ on } D_k \\ \vdots \\ 0 \text{ elsewhere on } X_0. \end{cases} \tag{25}$$

Clearly $\mu_c \pmod{Q(X_0)^\perp}$ is exactly $d_0S(c)$.
Then $d_0S(c)$, as a linear functional on $Q(X_0)$, is

$$\langle\psi, \mu_c\rangle = c_1\iint_{D_1}\psi + c_2\iint_{D_2}\psi + \ldots$$

$$= \text{the pairing of the } l^\infty\text{-sequence } c \text{ with the } l^1\text{-sequence } \theta(\psi).$$

This establishes the duality. The second statement now follows from the inverse function theorem for Banach spaces. This duality, for *arbitrary* Teichmüller spaces $T(G)$, is proved below.

Notice that in the finite-dimensional case the injectivity of θ (using d discs) was necessary and sufficient for the Schiffer parameters for the discs D_k to provide coordinates. Even for general $T(X_0)$ we see now that θ is injective if and only if each ψ in $Q(X_0)$ that vanishes at all the centres of D_k vanishes identically. This fits with the last assertion of Theorem 1.

Theorem 3 and the Bers embedding

The duality of Theorem 3 connects up with the Weil-Petersson pairing and the Bers embedding for *arbitrary* $T(G)$, *G a Fuchsian group with or without torsion. In this case the parametric discs* D_j *should be chosen within a fundamental domain for* G *in* U.

This general proof of $\theta^* = d_0 S$ is specially instructive since it hinges on a well-known reproducing formula which is ubiquitous in Teichmüller theory, namely,

$$\frac{12}{\pi}\iint_U \frac{\varphi(\zeta)\eta^2}{(z-\bar{\zeta})^4}\,d\xi d\eta = \varphi(z) \tag{26}$$

for any φ in $B(G)$ and any z in U.

Indeed, let Φ: $M(G) \to B(G)$ be Bers' natural projection. Its derivative at 0 is a map from $L^\infty(G)$ onto $B(G)$ given by:

$$d_0\Phi(\mu) = a\iint_U \frac{\overline{\mu(z)}}{(\bar{z}-\zeta)^4}\,dxdy \in B(G), \tag{27}$$

(a is a nonzero constant). See Bers [3] for these standard facts. (Since $B(G) = A(G)^*$ is a Banach space of holomorphic functions on U rather than on the lower half-plane the formulae here are (very) slightly modified.)

The tangent vector at X_0 in $T(X_0)$ corresponding to $d_0 S(c)$ is then $d_0\Phi(\mu_c)$, where μ_c is the Beltrami differential in (25) lifted to U as a G-invariant $(-1, 1)$ form, (still called μ_c). Thus,

$$d_0 S(c) = \varphi \in B(G), \text{ where } \varphi \text{ is } d_0\Phi(\mu_c).$$

Given any ψ in $A(G)$ we are required to show that the Weil-Petersson pairing (ψ, φ) is precisely

$$c_1\iint_{D_1}\psi + c_2\iint_{D_2}\psi + \ldots = \langle\psi, \mu_c\rangle.$$

But notice that

$$(\psi, \varphi) = \iint_{U/G}\psi(\zeta)\overline{\varphi(\zeta)\eta^2}\,d\xi d\eta$$

So the question is whether $\bar{\varphi}\eta^2$ and μ_c are equivalent linear functionals on $A(G)$. By Teichmüller's Lemma we know that this happens if and only if their difference is in the kernel of the map $d_0\Phi$. Thus we desire to check whether

$$d_0\Phi(\bar{\varphi}\eta^2) = d_0\Phi(\mu_c). \tag{28}$$

But the right side is, by definition, φ itself. The formula (27) for $d_0\Phi$ says therefore that (28) is indeed true (upto a fixed constant) because of the classic reproducing formula (26). We are through.

We conclude by observing that Shields and Williams [10] have proved that $A(1)$ is abstractly isomorphic to l^1. This fact is of course very relevant to the choice of Schiffer variation discs D_j for coordinatisation of universal Teichmüller space, $T(1)$.

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References

[1] Ahlfors L, *Lectures on quasiconformal mappings*, Van Nostrand, N.Y. (1966).
[2] Ahlfors L, The complex analytic structure of the space of closed Riemann surfaces, "*Analytic Functions*", Princeton University Press, Princeton, N.J. (1960).
[3] Bers L, *On Moduli of Riemann Surfaces*, Lecture notes, ETH, Zürich (1964)
[4] Earle C J, Reduced Teichmüller spaces, *Trans. Amer. Math. Soc.*, **126** (1967), 54–63
[5] Earle C J, On holomorphic cross-sections in Teichmüller spaces, *Duke Math. J.*, **36** (1969), 409–416
[6] Gardiner F P, Schiffer's interior variation and quasiconformal mappings, *Duke Math J.*, **42** (1975), 371–380
[7] Gardiner F P, The existence of Jenkins-Strebel differentials from Teichmüller theory, *Amer J. Math.* **99** (1975) 1097–1104
[8] Patt C, Variations of Teichmüller and Torelli surfaces, *J. d'Analyse Math.*, **11** (1963), 221–247
[9] Schiffer M and Spencer D *Functionals of Finite Riemann Surfaces*, Princeton U. P., Princeton, N. J. (1954)
[10] Shields A and Williams D, Bounded projections, duality and multipliers in spaces of analytic functions, *Trans. Amer. Math. Soc.* **162** (1971), 287–302

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Minimum degree of a graph and the existence of k-factors

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Abstract. Let G be a graph and k a positive integer such that (i) $k|V(G)|$is even; (ii) $\delta(G) \geqslant \frac{1}{2}[|V(G)|]$, and (iii) $|V(G)| \geqslant 4k-5$. Then G has a k-factor.

Keywords. Minimum degree; graph; k-factors.

All graphs considered are simple and finite. We refer the reader to [1] for standard graph theoretic terms not defined in this paper.

Let k be a positive integer. Then we say that a graph G has a k-factor if it has a k-regular spanning subgraph. If $X, Y \subseteq V(G)$, then $e(X, Y)$ denotes the number of edges having one end-vertex in X and the other in Y. Tutte [4] proved the following theorem.

Tutte's k-factor theorem: A graph G has a k-factor if and only if

$$q_G(D, S; k) + \sum_{x \in S} (k - d_{G\setminus D}(x)) \leqslant k|D|$$

for all $D, S \subseteq V(G), D \cap S = \varnothing$, where $q_G(D, S; k)$, denotes the number of components C of $(G\setminus D)\setminus S$ such that $k|V(C)| + e(C, S)$ is odd.

In the same paper he also showed that

$$q_G(D, S; k) + \sum_{x \in S} (k - d_{G\setminus D}(x)) - k|D| \equiv k|V(G)| \pmod 2. \tag{1}$$

The following theorem of Dirac [2] is well known.

THEOREM 1. Let G be a graph such that

(i) $|V(G)| \geqslant 3$

(ii) $\delta(G) \geqslant \dfrac{|V(G)|}{2}$

Then G has a Hamilton cycle.

We will state now a result due to Nash-Williams [3] which considerably sharpens Dirac's theorem for large graphs.

THEOREM 2. Let G be a graph such that $\delta(G) \geqslant \frac{1}{2}[|V(G)|]$. Then there exists a set of $[5(|V(G)|+\theta_{|V(G)|}+10)/224]$ edge-disjoint Hamiltonian cycles of G, where $\theta_{|V(G)|}$ denotes 0 if $|V(G)|$ is even and 1 if $|V(G)|$ is odd.

Theorem 2 has the following corollary.

COROLLARY 1. Let G be a graph. If $\delta(G) \geqslant \frac{1}{2}[|V(G)|]$ and $|V(G)|$ is even, then G has $[10(|V(G)|+10)/224]$ edge-disjoint 1-factors.

Proof. It follows immediately from theorem 2 if we consider a Hamilton cycle as a union of two edge-disjoint 1-factors.

The main theorem of this paper is as follows.

THEOREM 3. Let G be a graph and k a positive integer such that

(i) $k|V(G)|$ is even

(ii) $\delta(G) \geqslant \dfrac{|V(G)|}{2}$, and

(iii) $|V(G)| \geqslant 4k-5$.

Then G has a k-factor.

LEMMA 1. Let G be a graph and k a positive integer where $k|V(G)|$ is even. Suppose that there exists a pair of disjoint subsets D and S of $V(G)$ such that

$$q_G(D, S; k) + \sum_{x \in S} (k - d_{G\setminus D}(x)) > k|D|.$$

If $|D \cup S|$ is maximal with respect to the above inequality, then
(i) $|N_{G\setminus D}(u)| \geqslant k+1$ and $|N(u) \cap S| \leqslant k-1$ for every vertex u of $(G\setminus D)\setminus S$.
(ii) $|V(C)| \geqslant 3$ for every component C of $(G\setminus D)\setminus S$.

Proof. Let C be a component of $(G\setminus D)\setminus S$ and let u be a vertex of C. Suppose that $|N_{G\setminus D}(u)| \leqslant k$. Define $S' = S \cup \{u\}$.
Then since

$$q_G(D, S'; k) \geqslant q_G(D, S; k) - 1, \sum_{x \in S'} (k - d_{G\setminus D}(x)) \geqslant \sum_{x \in S} (k - d_{G\setminus D}(x))$$

and by using (1)

$$q_G(D, S'; k) + \sum_{x \in S'} (k - d_{G\setminus D}(x)) > k|D|$$

which contradicts the maximality of $|D \cup S|$. Thus $|N_{G\setminus D}(u)| \geqslant k+1$.

Now suppose that $|N(u) \cap S| \geqslant k$. Define $D' = D \cup \{u\}$. Then

$$q_G(D', S; k) \geqslant q_G(D, S; k) - 1$$

and

$$\sum_{x\in S}(k-d_{G\setminus D'}(x)) \geqslant \sum_{x\in S}(k-d_{G\setminus D}(x))+k.$$

Thus

$$q_G(D', S; k)+\sum_{x\in S}(k-d_{G\setminus D'}(x)) > k|D'|$$

which contradicts again the maximality of $|D\cup S|$. Thus $|N(u)\cap S| \leqslant k-1$. (ii) It follows immediately from (i) since $|N_{G\setminus D}(u)| - |N(u)\cap S| = |N(u)\cap V(C)|$ and thus $|N(u)\cap V(C)| \geqslant 2$.

LEMMA 2. If G is Hamiltonian, then for every non-empty proper subset X of $V(G)$,

$$\omega(G\setminus X) \leqslant |X|.$$

Proof. Let C be a Hamilton cycle of G. Then, for every non-empty proper subset X of $V(G)$

$$\omega(C\setminus X) \leqslant |X|.$$

But $C\setminus X$ is a spanning subgraph of $G\setminus X$ and so

$$\omega(G\setminus X) \leqslant \omega(C\setminus X).$$

Thus the lemma follows.

Proof of Theorem 3.
Suppose that G does not have a k-factor. Then by Tutte's k-factor theorem there exists $D, S\subseteq V(G)$, $D\cap S=\varnothing$ such that

$$q_G(D, S; k)+\sum_{x\in S}(k-d_{G\setminus D}(x)) > k|D|. \tag{2}$$

If $|D\cup S|$ is maximal with respect to (2), then by Lemma 1(ii), for every component C of $(G\setminus D)\setminus S$,

$$|V(C)| \geqslant 3. \tag{3}$$

Let $|V(G)|=n$, $\delta=\min_{x\in S}\{d_{G\setminus D}(x)\}$, $W=(G\setminus D)\setminus S$.

Then since $k|V(G)|$ is even, using (1), (2) becomes

$$q_G(D, S; k)+\sum_{x\in S}(k-d_{G\setminus D}(x)) \geqslant k|D|+2.$$

Thus

$$\omega((G\setminus D)\setminus S)+(k-\delta)|S| \geqslant k|D|+2. \tag{4}$$

If $S=\varnothing$,

$$\omega(G\setminus D) \geqslant k|D|+2. \tag{5}$$

But by theorem 1, G is Hamiltonian. So by lemma 2, $\omega(G\setminus X) \leqslant |X|$ for every non-empty subset X of $V(G)$, which contradicts (5). Thus $S \neq \varnothing$. Now if $\delta \geqslant k+1$,

$$\omega((G\setminus D)\setminus S) \geqslant k|D|+|S|+2. \tag{6}$$

Since G is Hamiltonian and by using again lemma 2, we get a contradiction. Hence $0 \leqslant \delta \leqslant k$.

Suppose that $0 \leqslant \delta \leqslant k-1$. Then from (4), we have

$$|S| \geqslant \frac{k|D|+2-\omega((G\setminus D)\setminus S)}{(k-\delta)}$$

and since $V(G) = D \cup S \cup V(W)$,

$$|S| \geqslant \frac{k(n-|S|-|V(W)|)+2-\omega((G\setminus D)\setminus S)}{(k-\delta)}.$$

Thus

$$|S| \geqslant \frac{kn+2-(k+1)|V(W)|}{(2k-\delta)}. \tag{7}$$

Let u be an element of S such that $d_{G\setminus D}(u) = \delta$. Then since

$$|D|+\delta \geqslant d(u) \geqslant \frac{n}{2}, \tag{8}$$

$$n-|S|-|V(W)|+\delta \geqslant \frac{n}{2}$$

hence

$$\delta+\frac{n}{2}-|V(W)| \geqslant |S|. \tag{9}$$

From (7) and (9)

$$\frac{n}{2}+\delta-|V(W)| \geqslant \frac{(kn+2)}{(2k-\delta)}-\frac{(k+1)}{(2k-\delta)}|V(W)|,$$

and since

$$\frac{(k+1)}{(2k-\delta)} \leqslant 1 \text{ for } \delta \leqslant k-1,\ \frac{n}{2}+\delta \geqslant \frac{kn+2}{2k-\delta}.$$

Thus

$$4\delta k - n\delta - 2\delta^2 \geqslant 4. \tag{10}$$

Clearly from (10), we have that $\delta \neq 0$, so

$$4k-2\delta-\frac{4}{\delta} \geqslant n. \tag{11}$$

But $4k-2\delta-\frac{4}{\delta} \leqslant 4k-6$ when $\delta \geqslant 1$. This contradicts the hypothesis that $n \geqslant 4k-5$.

Now suppose that $\delta = k$. Then (4) becomes

$$\omega((G\backslash D)\backslash S) \geqslant k|D|+2. \tag{12}$$

By (8),

$$|D| \geqslant \frac{n}{2}-k \text{ so } \omega((G\backslash D)\backslash S) \geqslant \frac{kn}{2}-k^2+2, \text{and by (3), } |V(W)| \geqslant \frac{3kn}{2}-3k^2+6.$$

Since $V(G) = D \cup S \cup V(W)$ and $S \neq \varnothing$

$$n = |D|+|S|+|V(W)| \geqslant \frac{3kn}{2}-3k^2+6+\frac{n}{2}-k+1.$$

Thus

$$\frac{6k^2+2k-14}{3k-1} \geqslant n.$$

But

$$4k-5 > \frac{6k^2+2k-14}{3k-1} \geqslant n. \tag{13}$$

So (13) contradicts the hypothesis that $|V(G)| \geqslant 4k-5$. Hence G has a k-factor.

The condition that $\delta(G) \geqslant \frac{1}{2}[|V(G)|]$ is clearly necessary because the complete bipartite graph $K_{m,m-2}$ does not possess a k-factor and $\delta(K_{m,m-2}) = m-2 = \frac{1}{2}[|V(G)|]-1$. The condition that $|V(G)| \geqslant 4k-5$ is also necessary. We will describe a graph G which does not have a k-factor although $|V(G)| = 4k-6$ and $\delta(G) = \frac{1}{2}[|V(G)|]$. We form G by joining every vertex of the complete graph $K_{2(k-2)}$ to all the vertices of $(k-1)$ copies of K_2. Let $X = \{u_1, v_1, u_2, v_2, \ldots, u_{k-1}, v_{k-1}\}$ where u_i, v_i denotes the two vertices of each K_2 and $Y = V(K_{2(k-2)})$.

Then if $D = Y$ and $S = X$

$$q_G(D, S; k) + \sum_{x \in S}(k - d_{G\backslash D}(x)) > k|D|$$

since $q_G(D, S; k) = 0$, $\sum_{x \in S}(k - d_{G\backslash D}(x)) = 2(k-1)^2$ and $k|D| = 2k(k-2)$. Thus G does not have a k-factor. Moreover $|V(G)| = |X|+|Y| = 2(k-1)+2(k-2) = 4k-6$ and $\delta(G) = 2k-3$.

References

[1] Bondy J A and Murty U S R, *Graph theory with applications*, North Holland, Amsterdam, 1976

[2] Dirac G A, Some theorems on abstract graphs, *Proc. London Math. Soc.* **2** (1952), 69–81

[3] Nash-Williams, C.St. J A, Edge-disjoint circuits in graphs with vertices of large valency. *Stud. Pure Math.*, Paper presented to R. Rado, Academic Press, London and New York, 1971

[4] Tutte W T, The factors of graphs, *Can. J. Math.* **4** (1952), 314–328

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Degenerated nonlinear hyperbolic equation with discontinuous coefficients

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Abstract. The partial differential equations with discontinuous coefficients have been extended in various directions by a number of authors [2], [3], [11], [13], [14], [16]. This paper deals with a mixed problem for a degenerated nonlinear hyperbolic equation with discontinuous coefficients.

Keywords. Partial differential equation; discontinuous coefficients; nonlinear hyperbolic equation.

1. Statement of the problem and notations

Let $Q = D \times [0, T]$ where D is a bounded domain in R^n with a smooth boundary Γ, Γ_1 separates D into two domains D_1 and D_2, $\gamma = \Gamma_1 \times [0, T]$, $S = \Gamma[0, T]$, $Q_1 = D_1[0, T]$ and $Q_2 = D_2[0, T]$.

Consider the hyperbolic equation

$$f(t)[u_{tt} - \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j}] + \sum_{i=1}^{n} b_i(x, t)u_{x_i} = F(x, t, u) \tag{1}$$

with initial conditions

$$u(x, 0) = u_t(x, 0) = 0, \tag{2}$$

boundary condition

$$u(x, t)|_S = 0, \tag{3}$$

and jumping conditions

$$u(x, t)|_{\gamma+0} = u(x, t)|_{\gamma-0},$$

$$K_1(x)\frac{du}{dN}\bigg|_{\gamma+0} - K_2(x)\frac{du}{dN}\bigg|_{\gamma-0} = k_3 u, \tag{4}$$

where $x = (x_1, \ldots, x_n)$, k_3 is a positive constant,

$$K(x) = \begin{cases} K_1(x), & x \in D_1 \\ K_2(x), & x \in D_2 \end{cases},$$

$$\frac{\mathrm{d}}{\mathrm{d}N} = \sum_{i,j=1}^{n} a_{ij} \cos(\nu, x_i) \frac{\partial}{\partial x_j},$$

and ν is the exterior unit normal of D_1.

For studying the problem (1)–(4) we have to define the following spaces; $W_2^1(Q) = W_2^1(Q_1) + W_2^1(Q_1)$ is a Sobolev space (see [1]); $E(Q)$ is a set of functions defined in Q and satisfy

$$\iint_Q \frac{\Phi^2(t, x)}{f(t)} < \infty.$$

We will refrain from writing the symbols $\mathrm{d}x$ and $\mathrm{d}x\mathrm{d}t$.

DEFINITION. The function $u(x, t)$ is called the solution of (1)–(4), if it satisfies

$$\iint_Q [K(x) u_{tt} \Phi + \sum_{i,j=1}^{n} a_{ij} u_{x_i} \Phi_{x_j} K(x) + \sum_{i,j=1}^{n} a_{ij} u_{x_i} K_{x_j} \Phi + \frac{1}{f(t)} \sum_{i=1}^{n} b_i u_{x_i} X\Phi - \frac{1}{f(t)} K(X) \Phi F(x, t, u)] + \int_0 \int_{\Gamma_1}^{T} k_3 u \Phi = 0, \tag{5}$$

and (2), (3) in L^2.

2. Assumption and results

THEOREM. Suppose that

(i) $f(t)$, $t > 0$ is continuous, differentiable and for small t satisfies

$$M_1 t^{\alpha} \leqslant f(t) \leqslant M_2 t^{\alpha}; \quad M_3 t^{\alpha - 1} \leqslant f'(t) \leqslant M_4 t^{\alpha - 1},$$

where $0 \leqslant \alpha < 1$ M_l, $l = 1, 2, \ldots$ are positive constants.

(ii) $\Sigma_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geqslant \lambda \Sigma_{i=1}^{n} \xi_i^2$, $\lambda > 0$ and $\partial b_i / \partial t$ is continuous in Q_1, Q_2.

(iii) $F(x, t, u)$ is measurable with respect to x, continuous with respect to t and has bounded derivatives F_t, F_u, in Q_1, Q_2 and $F(x, t, u) \leqslant p(x, t)u + q(x, t)$ where $p, q = 0(t^{\alpha + 3/2})$, $F_t = 0(t^{\alpha + 1/2})$, then there exists a unique weak solution to problem (1)–(4).

Proof. Let $v_r(x)$ be a linear independent system in $W_2^1(D)$, satisfy condition (4) and compact in $W_2^1(D)$. We seek the solution of problem (1)–(4) in the form (see [9]):

$$u_m(x, t) = \sum_{r=1}^{m} c_r(t) v_r(x).$$

Substitute $u(x, t)$ with $u_m(x, t)$ in (1), multiply by $v_s(x)$ and integrate over D; the result is the following system of ordinary differential equations:

$$f(t)\sum_{r=1}^{m} c_r''(t)\int_D Kv_r v_s + f(t)\sum_{r=1}^{m} c_r(t)\int_D \sum_{i,j=1}^{n} a_{ij}(v_r)_{x_i}(v_s)_{x_j}K$$
$$+f(t)\sum_{r=1}^{m} c_r(t)\int_{\Gamma_1} k_3 v_r v_s + \sum_{r=1}^{m} c_r(t)\int_D \sum_{i=1}^{n} b_i(v_r)_{x_i} v_s K$$
$$+f(t)\sum_{r=1}^{n} c_r(t)\cdot\int_D \sum_{i,j=1}^{n} a_{ij}(v_r)_{x_i} K_{x_j}\Phi$$
$$= \int_D KF\left(x, t\sum_{r=1}^{m} c_r v_r\right)v_s, \tag{6}$$

$s = (1, \ldots, m)$ with initial conditions,

$$c_r(0) = c_r'(0) = 0. \tag{7}$$

The existence of the solution to the problem (6)–(7) was proved (see [6], [12]).

Now, we will prove that $u_m(x, t)$ converges to $u(x, t)$ which is the weak solution to problem (1)–(4). To do so, we will find some important *a priori* estimates. This is usually where the hard work comes in. Many interesting estimates are discussed in [5]–[8] and [15]. Let $\eta_\varepsilon(x, t)$ be an infinitely differentiable function which tends to zero in $Q - Q_\varepsilon$, $\eta_\varepsilon = 1$ in $Q_{2\varepsilon}$ and $|\eta_\varepsilon| \leqslant 1$ in $Q_{2\varepsilon} - Q_\varepsilon$, where $\varepsilon > 0$. Substitute F with F_ε, where $F_\varepsilon = \eta_\varepsilon F$ in system (6) to get the following system:

$$f(t)\sum_{r=1}^{m} c_r''\int_D Kv_r v_s + f(t)\sum_{r=1}^{m} c_r\int_D \sum_{ij=1}^{n} a_{ij}(v_r)_{x_i}(v_s)_{x_j}K$$
$$+f(t)\sum_{r=1}^{m} c_r\int_{\Gamma_1} k_3 v_r v_s + f(t)\sum_{r=1}^{m} c_r\int_D \sum_{ij=1}^{n} a_{ij}(v_r)_{x_i}(K)_{x_j} v_s$$
$$+\sum_{r=1}^{m} c_r\int_D \sum_{i=1}^{n} b_i(v_r)_{x_i} v_s K = \int_D KF_\varepsilon\left(x, t, \sum_{r=1}^{m} c_r v_r\right)v_s. \tag{8}$$

For convenience, we will adopt the notation of u_m to be understood as $u_{m,\varepsilon}$. Multiply (8) by $c_{s,\varepsilon}'\, 1/f(t)t^{\beta+2\alpha}$, $0 < \beta < 1 - \alpha$; sum over s from 1 to m; integrate with respect to t from 0 to $T_1 < T$; to finally get:

$$\int_0^{T_1}\int_D \frac{1}{t^{\beta+2\alpha}} K(u_m)_{tt}(u_m)_t + \int_0^{T_1}\int_D \frac{1}{t^{\beta+2\alpha}} K\sum_{ij=1}^{n} a_{ij}(u_m)_{x_i}(u_m)_{x_j}$$
$$+\int_0^{T_1}\int_{\Gamma_1} \frac{1}{t^{\beta+2\alpha}} k_3 u_m(u_m)_t + \int_0^{T_1}\int_D \frac{1}{f(t)t^{\beta+2\alpha}} K\sum_{i=1}^{n} b_i(u_m)_{x_i}(u_m)_t$$
$$= \int_0^{T_1}\int_D \frac{1}{f(t)t^{\beta+2\alpha}} K(u_m)_t F_\varepsilon. \tag{9}$$

Integrate by parts to get:

$$\frac{1}{2}\int_D \frac{1}{t^{\beta+2\alpha}} K(u_m)_t^2\Big|_{t=T_1} + \frac{1}{2}\int_D \frac{1}{t^{\beta+2\alpha}} K \sum_{ij=1}^{n} a_{ij}(u_m)_{x_i}(u_m)_{x_j}\Big|_{t=T_1}$$
$$+\frac{1}{2}\int_{\Gamma_1} \frac{1}{t^{\beta+2\alpha}} k_3 u_m^2\Big|_{t=T_1} + \frac{B+2\alpha}{2}\int_0^{T_1}\int_D \frac{1}{t^{\beta+2\alpha+1}} K(u_m)_t^2$$
$$+\frac{\beta+2\alpha}{2}\int_0^{T_1}\int_D \frac{1}{t^{\beta+2\alpha+1}} K \sum_{ij=1}^{n} a_{ij}(u_m)_{x_i}(u_m)_{x_j}$$
$$+\frac{\beta+2\alpha}{2}\int_0^{T_1}\int_{\Gamma_1} \frac{1}{t^{\beta+2\alpha}} k_3 u_m^2$$
$$= \int_0^{T_1}\int_D \frac{1}{f(t)t^{\beta+2\alpha}} KF_3(u_m)_t - \int_0^{T_1}\int_D \frac{1}{f(t)t^{\beta+2\alpha}} \sum_{i=1}^{n} b_i K(u_m)_{x_i}(u_m)_t$$
$$-\int_0^{T_1}\int_D \frac{1}{t^{\beta+2\alpha}} \sum_{ij=1}^{n} a_{ij}(u_m)_{x_i} K_{x_j}(u_m)_t. \tag{10}$$

Estimate the first term on the right hand side using the inequality $|ab| \leqslant 1/2\,a^2 + 1/2b^2$ and the theorem conditions to obtain:

$$\left|\int_0^{T_1}\int_D \frac{1}{t^{\beta+2\alpha}f(t)} KF_\varepsilon(u_m)_t\right|$$
$$\leqslant \left|\int_0^{T_1}\int_D \frac{1}{t^{\beta+2\alpha}f(t)} Pu_m(u_m)_t\right| + \int_0^{T_1}\int_D \frac{1}{f(t)t^{\beta+2\alpha}}(u_m)_t q\Big|$$
$$\leqslant \frac{1}{2}\int_0^{T_1}\frac{1}{f(t)t^{\beta+2\alpha}} p^2 u_m^2 + \int_0^{T_1}\int_D \frac{1}{f(t)t^{\beta+2\alpha}}(u_m)_t^2$$
$$+\frac{1}{2}\int_0^{T_1}\int_D \frac{1}{t^{\beta+2\alpha}} q^2. \tag{11}$$

From the above, it is easy to arrive at

$$\int_0^{T_1}\int_D \frac{1}{f(t)t^{\beta+2\alpha}} p^2 u_m^2 \leqslant C\frac{T_1^5}{4(2+\beta+2\alpha)}\int_0^{T_1}\int_D \frac{1}{t^{\beta+2\alpha+1}}(u_m)_t^2, \tag{12}$$

where C is a constant independent of m, ε. Similarly, we can estimate the remaining terms in (10), and therefore the inequality (10) can be written as:

$$\int_0^{T_1}\int_D \frac{A}{t^{1+\beta+2\alpha}}(u_m)_t^2 + \int_0^{T_1}\int_D \frac{B}{t^{1+\beta+2\alpha}}\sum_{i=1}^{n}(u_m)_{x_i}^2 + \int_0^{T}\int_{\Gamma_1}\frac{1}{t^{1+\beta+2\alpha}}u^2$$
$$\leqslant \int_0^{T_1}\int_D \frac{1}{f(t)t^{\beta+2\alpha}} q^2, \tag{13}$$

where

$$A = \frac{\beta+2\alpha}{2} - \frac{\text{cnt}}{2\phi} - \frac{\text{cnt}}{2} - \frac{t}{\phi} - \frac{CT_1^5}{8(2+\beta+2\alpha)},$$

and $$B = \frac{\beta + 2\alpha}{2}\lambda - \frac{Ct}{2\phi} - \frac{Ct}{2},$$

choose T_1 sufficiently small such that A and B remain positive. From the integral on the right side in (13), together with the condition on $q(x, t)$ we find (see [4]):

$$\begin{aligned}
&\int_0^{T_1}\int_D \frac{1}{t^{1+\beta+2\alpha}}(u_m)_t^2 < C,\\
&\int_0^{T_1}\int_D \frac{1}{t^{1+\beta+2\alpha}}\sum_{i=1}^{n}(u_m)_{x_i}^2 < C,\\
&\int_0^{T_1}\int_D \frac{1}{t^{1+\beta+2\alpha}}u_m^2 < C.
\end{aligned} \tag{14}$$

Multiply (8) by $1/f(t)$; differentiate with respect to t and multiply the result by $C''_{s,\varepsilon}$ $1/t^{\beta}$; sum over s from 1 to m and integrate with respect to t from 0 to $T_1 < T$ to get:

$$\begin{aligned}
&\int_0^{T_1}\int_D \frac{1}{t^{\beta}}K(u_m)_{tt}(u_m)_t + \int_0^{T_1}\int_D \frac{1}{t^{\beta}}K\sum_{ij=1}^{n}(u_m)_{x_i}(u_m)_{x_j}\\
&+\int_0^{T_1}\int_{\Gamma_1}\frac{1}{t^{\beta}}k_3(u_m)_t(u_m)_{tt} + \int_0^{T_1}\int_D \frac{1}{t^{\beta}f(t)}K\sum_{i=1}^{n}(b_i)_t(u_m)_{x_j}(u_m)_t\\
&+\int_0^{T_1}\int_D \frac{1}{t^{\beta}f(t)}K\sum_{i=1}^{n}b_i u_m(u_m)_{tt} - \int_0^{T_1}\int_D \frac{f'(t)}{t^{\beta}f^2(t)}K\sum_{i=1}^{n}b_i(u_m)_{x_i}(u_m)_{tt}\\
&+\int_0^{T_1}\int_D \frac{1}{t^{\beta}}\sum_{ij=1}^{n}a_{ij}(K)_{x_j}(u_m)_{tx_i}(u_m)_{tt}\\
&\quad = \int_0^{T_1}\int_D \frac{1}{t^{\beta}f(t)}(F_\varepsilon)_t K(u_m)_{tt} + \int_0^{T_1}\int_D \frac{1}{t^{\beta}f(t)}(F_\varepsilon)_{u_m}(u_m)_t(u_m)_{tt}\\
&\quad -\int_0^{T_1}\int_D \frac{f'(t)}{t^{\beta}f^2(t)}KF_\varepsilon(u_m)_{tt}.
\end{aligned} \tag{15}$$

From (15), the fact that $|F_\varepsilon| \leqslant |F|$ and the theorem conditions, we get the next *a priori* estimates:

$$\begin{aligned}
&\int_0^{T_1}\int_D \frac{1}{t^{1+\beta}}(u_m)_{tt}^2 < C,\\
&\int_0^{T_1}\int_D \frac{1}{t^{1+\beta}}(u_m)_{tx_i}^2 < C,\\
&\int_0^{T_1}\int_D \frac{1}{t^{1+\beta}}(u_m)_t^2 < C.
\end{aligned} \tag{16}$$

Using the *a priori* estimates (14) and (16) we can choose subsequences $(u_{mk})_t$, $(u_{mk})_{x_i}$, $(u_{mk})_{tt}$, $(u_{mk})_{tx_i}$ converges to u_t, u_{xi}, u_{tt}, u_{tx_i} in L_2. Regarding the embedding theorem (see [1]) u_m converges strongly to u.

To prove the uniqueness, assume u and $\bar{u}$ are two different solutions and $w = u - \bar{u}$. Using the *a priori* estimates in (14) and (16), we clearly get $w = 0$, i.e., $u = \bar{u}$. Thus, the existence and uniqueness have been proved.

REMARK. Similarly, we can prove the existence theorem for problems (1)–(4), in case of $F(x, t, u) = a(x, t)|u|$ where $\sigma \leqslant n + 2/n - 1$ in place of condition 3 (see [10]).

References

[1] Adams R A, *Sobolev spaces*, Academic Press, New York, 1975.

[2] Chandirov G and Rzoev E, Approximation of solutions of nonregular mixed problems for equations of parabolic type with discontinuous and constant coefficients by solutions of regular problem, *Az. Gos. Univ. Baku, USSR.* (1981) 102–111

[3] De Simon L, Inhomogeneous first order linear partial differential equations with coefficients and right hand side discontinuous, *Boll. Un. Mat. Ital. B* (6) **1** (1982) 729–742

[4] Duvaut G, *Inequalities in mechanics and physics*, Springer-Verlag, Berlin, 1976

[5] Egarov E A, Boundary value problem for elliptic-parabolic equations, *Sibirsk, Mat. Z.* **17** (1976) 686–691

[6] Hartman P, *Ordinary differential equations*, John Wiley, New York, 1964

[7] John F and Klainerman S, Almost global existence to nonlinear wave equation in three space dimensions, *Commun. Pure Appl. Math.*, **37** (1984)

[8] Klainerman S, Solutions to quasilinear wave equations in three space dimensions, *Commun. Pure Appl. Math.* **36** (1983) 325–344

[9] Krasnosclskii M A, *Topological methods in the theory of nonlinear integral equations*, Macmillans, New York, 1964

[10] Nirenberg L, Variational and topological methods in nonlinear problems, *Bull. Am. Math. Soc.* **4** (1981) 267–302

[11] Oleinik O A, Boundary value problem for linear parabolic equation with discontinuous coefficients, *Uzv. Akad. Nauk SSR, Scr. Mat.*, **25** (1961) 9–20

[12] Petrovski T G, *Ordinary differential equations*, Prentice Hall, New Jersey, 1966

[13] Romaskevich L P, Classical solutions of second order quasilinear degenerate elliptic equations with discontinuous coefficients, *Usp. Mat. Nauk* **36** (1980) 205–216

[14] Shen Y T, Global boundedness of weak solution of quasilinear elliptic equation of order two with strongly singular coefficients, *J. China Univ. Sci. Tech.* **11** (1981) 24–31

[15] Sigillito V, *Explicit a priori inequalities with applications to boundary value problem*, Pitman, London, 1977

[16] Venturino M, The membership in $H^3(\Omega)$ of the solutions of a class of elliptic equations with discontinuous coefficients, *Boll. Un. Mat. Ital. Suppl.* (1980) 197–218

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A three-layer asymptotic analysis of turbulent channel flow

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Abstract. In the classical theory for large-Reynolds number fully developed channel flow, the solutions obtained by asymptotic-expansion techniques for the outer Karman defect layer and the inner Prandtl wall layer are demonstrated to match through the introduction of an intermediate layer based on a general intermediate limit. From an examination of the results for this general intermediate layer, the distinguished intermediate limit and the corresponding intermediate layer for which the turbulent and laminar contributions to the difference of the Reynolds stress from the wall stress are of the same order of magnitude are identified. The thickness of this distinguished intermediate layer is found to be of the order of the geometric mean of the thicknesses of the outer and inner layers

Keywords. Asymptotic analysis; turbulent channel flour; eddy-viscosity closure.

1. Introduction

The present paper demonstrates that, in terms of the classical theory, for turbulent channel (or pipe) flow, there is a *distinguished* intermediate between the outer defect layer and the inner wall layer. With this demonstration, it follows that this (new) three-layer theory modifies and/or extends the (old) two-layer one.

A review and extension of the asymptotic analysis of turbulent channel flow is presented in §2. The non-dimensional boundary-value problem, without the adoption of closure, is introduced, and the basic formulations of the boundary-value problem in terms of outer-layer and inner-layer variables are developed. The asymptotic representations of the Reynolds stress and the mean velocity for the outer and inner layers, as determined from the solutions of the pertinent boundary-value problem are given, and their behaviors in a region intermediate to the defect and wall layers are, then, considered.

If $R_\tau = u^* h^*/\nu^*$ is the turbulent Reynolds number, much greater than unity, with h^* the half-depth of the channel, u_τ^* the friction velocity, and ν^* the kinematic viscosity coefficient, and if $z = z^*/h^*$ and $\zeta = u_\tau^* z^*/\nu^* = R_\tau z$ are the outer-layer and inner-layer spatial coordinates, respectively, with z^* the distance from the wall, then λ, the *general* intermediate-layer spatial coordinate, is defined by $\lambda = z/\phi(R_\tau) = \zeta/R_\tau\phi(R_\tau)$, with $\phi(R_\tau) \to 0$, $R_\tau\phi(R_\tau) \to \infty$ as $R_\tau \to \infty$. In this paper, attention is directed to the *particular* (or *distinguished*) intermediate-layer spatial-coordinate case of $\eta = \lambda$, with $\phi(R_\tau) = 1/R_\tau^{1/2} \to 0$ and/or $R_\tau\phi(R_\tau) = R_\tau^{1/2} \to \infty$, such that the defect-layer and

the wall-layer variables satisfy $z = \eta/R_\tau^{1/2} \to 0$ and $\zeta = R_\tau^{1/2}\,\eta \to \infty$, respectively, as $R_\tau \to \infty$, with η fixed. This distinguished intermediate-layer spatial coordinate is the one whose introduction produces an intermediate layer in which the turbulent and laminar contributions to the difference of the Reynolds stress from the wall stress are of the same order of magnitude. If the outer-layer and inner-layer asymptotic solutions for the Reynolds stress and the mean velocity match in a general intermediate layer, then these solutions also match in the distinguished intermediate layer introduced. The solutions for the Reynolds stress and the mean velocity in this distinguished intermediate layer, as determined from the outer-layer and/or inner-layer solutions for these flow quantities, are presented.

The results obtained for this distinguished intermediate layer should be considered for the development of an intermediate-layer distinguished limit. The formulation and solutions for this intermediate-layer distinguished limit are presented. The analyses of the outer-layer and inner-layer distinguished limits, complemented by the analysis of the intermediate-layer distinguished limit, complete the three-layer asymptotic analysis of turbulent channel flow.

This intermediate-layer asymptotic analysis yields quantitative results for the magnitude and location of the maximum of the Reynolds stress that is achieved within this layer. In addition to the behaviour of the Reynolds stress, which, in effect, defines this intermediate layer, attention is directed to the behaviour of the mean velocity in this layer. It is demonstrated here that, to leading order of approximation, this velocity is a constant equal to one-half the value of the channel center-line velocity, and that (not surprisingly) the first correction to this value is of the order of u_τ and is functionally logarithmic. The location (to leading order of approximation) of the "half-velocity point" within this intermediate layer is found in the analysis of §2.

In §3, an eddy-viscosity closure model, namely that of Bush and Fendell [4], is adopted, and the solutions for the outer and inner layers and particularly those for the distinguished intermediate layer for this closure model are determined. The heretofore undetermined constants of the asymptotic analysis of §2 are evaluated in terms of the parameters of this closure model. For the purpose of comparison, the solution for the mixing-length-theory eddy-viscosity closure model of Bush and Fendell [5] for the Reynolds stress in this intermediate layer is presented also.

The existence of an intermediate layer in turbulent shear flows has been considered previously by Long and Chen [7] and by Afzal [2, 3]. Long and Chen proposed a mesolayer (or intermediate-layer) theory, in which the inner-wall-layer and intermediate-layer expansions are developed as developed as a composite expansion that is matched with the outer-defect-layer expansion. They claim that, since the intermediate layer "intrudes" between the outer and inner regions preventing the commonly assumed overlap and the consequent logarithmic profile, their theory replaces the classical two-layer theory. Afzal analyzed the three layers separately and matched the resulting asymptotic solutions in two overlap domains to demonstrate the possibility of two logarithmic regions rather than the one logarithmic region of classical theory.

2. Asymptotic analysis

The non-dimensional boundary-value problem for the steady two-dimensional turbulent flow of a fluid of constant density and viscosity (i.e., $\rho^*, \nu^* = \text{constant}$) in a channel is

$$\tau + \frac{1}{R}\frac{du}{dz} = u_\tau^2 (1-z) \qquad (0 \leqslant z \leqslant 1); \tag{1}$$

$$u \to 1, \frac{du}{dz} \to 0, \tau \to 0 \text{ as } z \to 1, \tag{2a}$$

$$u, \tau \to 0 \text{ as } z \to 0. \tag{2b}$$

The independent variable is $z = z^*/h^*$, the normal coordinate, with h^* the half-depth of the channel ($z = 1$ at the channel centreline, $z = 0$ at the channel wall); the dependent variables are $u = u^*/u_0^* = u(z; R)$, the mean velocity in the axial direction, with u_0^* the mean axial velocity at the centreline, and $\tau = \tau^*/\rho^* u_0^{*2} = (-\overline{u^{*\prime} v^{*\prime}})/u_0^{*2} = \tau(z; R)$, the turbulent shear stress. The parameters are $R = u_0^* h^*/\nu^*$, the Reynolds number, taken to be much greater than unity; and $u_\tau = u_\tau^*/u_0^* = [\nu^* (du^*/dz^*)_{z^*=0}]^{1/2}/u_0^* = u_\tau(R)$, the friction velocity, effectively an eigenvalue (to be determined), less than unity. Specifically, in what follows, it is taken that

$$u_\tau \to 0 \text{ as } R \to \infty, \tag{3a}$$

$$R_\tau = R u_\tau = u_\tau^* h^*/\nu^* \to \infty \text{ as } R \to \infty. \tag{3b}$$

For $R \to \infty$, the classical theory of turbulent channel flow divides the flow into two regions: an outer (or defect) layer and an inner (or wall) layer. The outer layer is the thicker layer, about the centreline and not immediately adjacent to the wall, in which the velocity undergoes only a small perturbation from its centreline value (the Karman velocity defect law), and the laminar stress is small relative to the turbulent stress. The inner layer is the thinner region, adjacent to the wall, in which the velocity is only a small fraction of its centreline value (the Prandtl universal law of the wall), and the turbulent and laminar stress are of comparable magnitude.

Asymptotic analyses of the defect layer and the wall layer follow.

For the defect layer, it is taken that the appropriate spatial variable is z, and the velocity and Reynolds stress variables are

$$u(z; R) = 1 - u_\tau(R_\tau)\, f(z; R_\tau), \tag{4a}$$

$$\tau(z; R) = u_\tau^2(R_\tau) t(z; R_\tau). \tag{4b}$$

The outer limit is defined as z fixed for $R_\tau \to \infty$. Thus, based upon substitution of (4) into (1) and (2a), the boundary-value problem for the defect layer ($0 < z \leqslant 1$) is

$$t - \frac{1}{R_\tau}\frac{df}{dz} = (1-z); \tag{5}$$

$$f, \frac{\mathrm{d}f}{\mathrm{d}z}, t \to 0 \text{ as } z \to 1. \tag{6}$$

An examination of (5) suggests the following asymptotic expansions for the defect-layer dependent variables:

$$f(z; R_\tau) \cong f_0(z) + \frac{1}{R_\tau} f_1(z) + \frac{1}{R_\tau^2} f_2(z) + \ldots, \tag{7a}$$

$$t(z; R_\tau) \cong t_0(z) + \frac{1}{R_\tau} t_1(z) + \frac{1}{R_\tau^2} t_2(z) + \ldots. \tag{7b}$$

Introduction of these expansions into the defect-layer differential equation produces the sequence

$$t_0 = (1-z), t_1 - \frac{\mathrm{d}f_0}{\mathrm{d}z} = 0, t_2 - \frac{\mathrm{d}f_2}{\mathrm{d}z} = 0, \ldots. \tag{8}$$

Thus, the leading-order approximation for the defect-layer turbulent stress is $t_0(z) = (1-z)$. This approximation cannot satisfy the pertinent boundary condition of (2b), as the formulation of (4) and the expansions of (7) are not valid at the wall.

To supplement the defect layer, the wall layer is introduced. For the wall layer, it is taken that the appropriate spatial variable is $\zeta = u_\tau^* z^*/\nu^* = R_\tau z$, and the velocity and Reynolds stress variables are

$$u(z; R) = u_\tau(R_\tau) g(\zeta; R_\tau), \tag{9a}$$

$$\tau(z; R) = u_\tau^2(R_\tau) s(\zeta; R_\tau), \tag{9b}$$

with the inner limit defined as ζ fixed for $R_\tau \to \infty$. Based upon substitution of (9) into (1) and (2b), the boundary-value problem for the wall layer ($0 \leqslant \zeta < \infty$) is

$$s + \frac{\mathrm{d}g}{\mathrm{d}\zeta} = \left(1 - \frac{1}{R_\tau}\zeta\right); \tag{10}$$

$$g, s \to 0 \text{ as } \zeta \to 0. \tag{11}$$

An examination of (10) suggests the following asymptotic expansions for the wall-layer dependent variables:

$$g(\zeta; R_\tau) \cong g_0(\zeta) + \frac{1}{R_\tau} g_1(\zeta) + \frac{1}{R_\tau^2} g_2(\zeta) + \ldots, \tag{12a}$$

$$s(\zeta; R_\tau) \cong s_0(\zeta) + \frac{1}{R_\tau} s_1(\zeta) + \frac{1}{R_\tau^2} s_2(\zeta) + \ldots. \tag{12b}$$

Introduction of these expansions into the wall-layer differential equation produces the sequence

$$s_0 + \frac{\mathrm{d}g_0}{\mathrm{d}\zeta} = 1, s_1 + \frac{\mathrm{d}g_1}{\mathrm{d}\zeta} = -\zeta, s_2 + \frac{\mathrm{d}g_2}{\mathrm{d}\zeta} = 0, \ldots. \tag{13}$$

It is noted that the leading-order equation of (13) corresponds to constant total stress.

Based on the Karman velocity defect law and the Prandtl law of the wall, through an extension of Millikan's argument (cf., e.g., Afzal [1]), the outer expansion as $z \to 0$ and the inner expansion is $\zeta \to \infty$ for the velocity profile are, respectively,

$$\begin{aligned} & u(z; R) = u(z; R_\tau) \\ & \sim 1 - u_\tau(R_\tau)\Big[(-A_{01} \log z + A_0 + W_0 z + \tfrac{1}{2} C_0 z^2 + \ldots) \\ & + \frac{1}{R_\tau}\left(\frac{J_1}{z} - A_{11} \log z + A_1 + W_1 z + \ldots\right) \\ & + \frac{1}{R_\tau^2}\left(\frac{L_2}{2z^2} + \frac{J_2}{z} - A_{21} \log z + A_2 + \ldots\right) + \ldots\Big]; \end{aligned} \tag{14a}$$

$$\begin{aligned} & u(z; R) = u(\zeta; R_\tau) \\ & \sim u_\tau(R_\tau)\Big[\left(B_{01} \log \zeta + B_0 + \frac{V_0}{\zeta} + \frac{D_0}{2\zeta^2} + \ldots\right) + \frac{1}{R_\tau}\left(H_1 \zeta + B_{11} \log \zeta + B_1\right. \\ & \left. + \frac{V_1}{\zeta} + \ldots\right) + \frac{1}{R_\tau^2}\left(\frac{1}{2} E_2 \zeta^2 + H_2 \zeta + B_{21} \log \zeta + B_2 + \ldots\right) + \ldots\Big]. \end{aligned} \tag{14b}$$

It can be demonstrated that (14a) and (14b) match in a region between the defect and wall layers through the introduction of an intermediate variable λ, defined by

$$\lambda = z/\phi(R_\tau) = \zeta/R_\tau \phi(R_\tau) \text{ fixed,}$$

$$\text{with } \phi(R_\tau) \to 0,\ R_\tau \phi(R_\tau) \to \infty \text{ as } R_\tau \to \infty, \tag{15a}$$

such that

$$z = \phi(R_\tau)\lambda \to 0, \quad \zeta = R_\tau \phi(R_\tau)\lambda \to \infty. \tag{15b}$$

This matching requires that

$$\begin{aligned} & B_{01} = A_{01} = 1/K_0,\ H_1 = -W_0,\ V_0 = -J_1,\ E_2 = C_0, \\ & B_{11} = A_{11} = 1/K_1,\ D_0 = -L_2,\ H_2 = -W_1,\ V_1 = -J_2, \ldots. \end{aligned} \tag{16a}$$

and that the following skin friction law be satisfied:

$$\begin{aligned} & u_\tau(R_\tau)\left[\left\{\frac{1}{K_0} \log R_\tau + (A_0 + B_0)\right\} + \frac{1}{R_\tau}\left\{\frac{1}{K_1} \log R_\tau + (A_1 + B_1)\right\} + \ldots\right] \\ & = 1: \\ & u_\tau(R_\tau) = \frac{K_0}{\log R_\tau}\left[\left\{1 + \frac{K_0(A_0 + B_0)}{\log R_\tau}\right\} + \frac{K_0/K_1}{R_\tau}\left\{1 + \frac{K_1(A_1 + B_1)}{\log R_\tau}\right\}\right. \\ & \left. + \ldots\right]^{-1}. \end{aligned} \tag{16b}$$

The corresponding expressions for the Reynolds stress profile are then determined to be

$$\tau(z;R)=\tau(z;R_\tau)$$
$$\sim u_\tau^2(R_\tau)\left[(1-z)+\frac{1}{R_\tau}\left(-\frac{A_{01}}{z}+W_0+C_0z+\ldots\right)+\frac{1}{R_\tau^2}\left(-\frac{J_1}{z^2}-\frac{A_{11}}{z}+\ldots\right)+\frac{1}{R_\tau^3}\left(-\frac{L_2}{z^3}-\frac{J_2}{z^2}+\ldots\right)+\ldots\right]; \tag{17a}$$

$$\tau(z;R)=\tau(\zeta;R_\tau)$$
$$\sim u_\tau^2(R_\tau)\left[\left(1-\frac{B_{01}}{\zeta}+\frac{V_0}{\zeta^2}+\frac{D_0}{\zeta^3}+\ldots\right)+\frac{1}{R_\tau}\left(-\zeta-H_1-\frac{B_{11}}{\zeta}+\ldots\right)+\frac{1}{R_\tau^2}\left(-E_2\zeta-H_2+\ldots\right)+\ldots\right]. \tag{17b}$$

Subject to (15) and (16), matching again follows.

From (17), it is seen that the leading-order turbulent contribution to the departure of the turbulent stress from $\tau_w(R_\tau)=u_\tau^2(R_\tau)$ is $z=\zeta/R_\tau\to 0$, whereas the leading-order laminar contribution to this departure is $1/K_0\zeta=1/K_0R_\tau z\to 0$. These turbulent and laminar contributions are seen to be of the same order of magnitude (Afzal [2]) for

$$\eta=R_\tau^{1/2}z=\zeta/R_\tau^{1/2}\ \text{fixed}. \tag{18}$$

The introduction of (18) into (17) yields

$$\tau(z;R)=\tau(\eta;R_\tau)$$
$$\cong u_\tau^2(R_\tau)\left[1-\frac{1}{R_\tau^{1/2}}\left(\eta+\frac{1}{k_0\eta}\right)+\frac{1}{R_\tau}\left(W_0-\frac{J_1}{\eta^2}\right)+\frac{1}{R_\tau^{3/2}}\left(C_0\eta-\frac{1}{K_1\eta}-\frac{L_2}{\eta^3}\right)+\ldots\right]. \tag{19}$$

Further, introduction of (18) into (14) produces

$$u(z;R)=u(\eta;R_\tau)$$
$$\cong\frac{1}{2}+u_\tau(R_\tau)\left[\left\{\frac{1}{K_0}\log\eta+\frac{1}{2}(B_0-A_0)\right\}-\frac{1}{R_\tau^{1/2}}\left\{W_0\eta+\frac{J_1}{\eta}\right\}+\frac{1}{R_\tau}\left\{\frac{1}{K_1}\log\eta+\frac{1}{2}(B_1-A_1)-\frac{1}{2}\left(C_0\eta^2+\frac{L_2}{\eta^2}\right)\right\}-\frac{1}{R_\tau^{3/2}}\left\{W_1\eta+\frac{J_2}{\eta}\right\}+\ldots\right]. \tag{20}$$

Thus, $\eta=\lambda$, with $\phi(R_\tau)=1/R_\tau^{1/2}$, represents a distinguished intermediate variable, whose introduction produces an intermediate layer, in which the turbulent and laminar contributions to the departure of the turbulent stress from u_τ^2 are of the same order of magnitude. These results indicate that a three-layer picture, rather than a two-layer

picture, of turbulent channel flow holds. That is, in addition to the defect-layer and wall-layer limits, there exists an intermediate limit. Based on (18)–(20), for the intermediate layer, it is taken that the appropriate spatial variable is

$$\eta = R_\tau^{1/2} z = \zeta / R_\tau^{1/2} = z^*/(h^* \nu^*/u_\tau^*)^{1/2} \text{ fixed;} \tag{21}$$

and the velocity and Reynolds stress variables are

$$u(z; R) = \frac{1}{2} + u_\tau(R_\tau) U(\eta; R_\tau), \tag{22a}$$

$$\tau(z; R) = u_\tau^2(R_\tau)\left[1 - \frac{1}{R_\tau^{1/2}} T(\eta; R_\tau)\right]. \tag{22b}$$

The intermediate limit is defined as η fixed for $R_\tau \to \infty$. Substitution of (21) and (22) into (1) produces the following differential equation for the intermediate layer ($0 < \eta < \infty$):

$$T - \frac{dU}{d\eta} = \eta. \tag{23}$$

For this layer, then, all three terms are of equal order, and this intermediate limit is identified as a distinguished limit.

It is taken that the appropriate asymptotic expansions for the intermediate-layer dependent variables are

$$U(\eta; R_\tau) \cong U_0(\eta) + \frac{1}{R_\tau^{1/2}} U_1(\eta) + \frac{1}{R_\tau} U_2(\eta) + \ldots, \tag{24a}$$

$$T(\eta; R_\tau) \cong T_0(\eta) + \frac{1}{R_\tau^{1/2}} T_1(\eta) + \frac{1}{R_\tau} T_2(\eta) + \ldots. \tag{24b}$$

Introduction of these expansions into the intermediate-layer differential equation produces the sequence

$$T_0 - \frac{dU_0}{d\eta} = \eta,\ T_1 - \frac{dU_1}{d\eta} = 0,\ T_2 - \frac{dU_2}{d\eta} = 0, \ldots. \tag{25}$$

To the orders of approximation considered, the solutions for the velocity and Reynolds stress functions, indicated by (19) and (20), are

$$U_0 = \frac{1}{K_0} \log \eta + \frac{1}{2}(B_0 - A_0),\ T_0 = \left(\eta + \frac{1}{K_0 \eta}\right); \tag{26a}$$

$$U_1 = -\left(W_0 \eta + \frac{J_1}{\eta}\right),\ T_1 = -\left(W_0 - \frac{J_1}{\eta^2}\right); \tag{26b}$$

$$U_2 = \frac{1}{K_2} \log \eta + \frac{1}{2}(B_1 - A_1) - \frac{1}{2}\left(C_0 \eta^2 + \frac{L_2}{\eta^2}\right),$$

$$T_2 = -\left(C_0 \eta - \frac{1}{K_1 \eta} - \frac{L_2}{\eta^3}\right). \tag{26c}$$

It can be shown that these solutions for the intermediate-layer dependent variables match directly to both the defect-layer and wall-layer solutions for these variables.

From (22) and (26), it follows that the velocity and turbulent stress for the intermediate layer are

$$u(z;R) = u(\eta;R_\tau)$$

$$\cong \frac{1}{2} + u_\tau(R_\tau)\left[\left\{\frac{1}{K_0}\log\eta + \frac{1}{2}(B_0 - A_0)\right\} + \ldots\right], \tag{27a}$$

$$\tau(z;R) = \tau(\eta;R_\tau)$$

$$\cong u_\tau^2(R_\tau)\left[1 - \frac{1}{R_\tau^{1/2}}\left\{\eta + \frac{1}{K_0\eta}\right\} + \ldots\right], \tag{27b}$$

where

$$u_\tau(R_\tau) \cong \frac{K_0}{\log R_\tau}\left[\left\{1 - \frac{K_0(B_0 + A_0)}{\log R_\tau}\right\} + \ldots\right]. \tag{28}$$

To the order of approximation considered in (27a), it is seen that

$$u \to \tfrac{1}{2} \text{ as } \eta \to \eta_{\frac{1}{2}} = \exp\{-\tfrac{1}{2}K_0(B_0 - A_0)\}. \tag{29}$$

For $K_0 \cong 0.41$ and $B_0 \cong 5.0$, $A_0 \cong 0.65$, $\eta_{1/2} \cong 0.41$. Further, from (27b) it is seen that

$$\tau \to \tau_{\max} \cong u_\tau^2\left[1 - \frac{2}{K_0^{1/2}R_\tau^{1/2}} + \ldots\right] \text{ as } \eta \to \eta_{\max} = \frac{1}{K_0^{1/2}}. \tag{30}$$

For $K_0 \cong 0.41, \eta_{\max} \cong 1.56$. A study of the experimental data for pipe and channel flows (Afzal [3]) suggests the values $\eta_{\frac{1}{2}} \cong 0.44$ and $\eta_{\max} \cong 1.85$.

3. Eddy-viscosity closure

With the adoption of the eddy-viscosity closure model, defined by

$$\tau = \frac{\varepsilon}{R}\frac{du}{dz}, \tag{31}$$

where $\varepsilon = \varepsilon^*/\nu^* = \varepsilon(z;R)$ is the kinematic eddy viscosity, the nondimensional boundary-value problem of (1) and (2) becomes

$$\frac{(1+\varepsilon)}{R}\frac{du}{dz} = u_\tau^2(1-z); \tag{32}$$

$$u \to 1, \frac{du}{dz} \to 0 \quad \text{as } z \to 1, \tag{33a}$$

$$u, \varepsilon \to 0 \quad \text{as } z \to 0. \tag{33b}$$

In order to solve this boundary-value problem, the kinematic eddy viscosity is specified [4] as

$$\varepsilon(z;R) = \varepsilon(z;R_\tau) = R_\tau M(z)N(\zeta), \tag{34}$$

where $\zeta = R_\tau z$. The functions $M(z)$ and $N(\zeta)$ have the following asymptotic behaviours:

$$M(z) = \kappa z\, M_0(z) \to \kappa z\,(1+\kappa_2 z^2 + \ldots) \to 0 \quad \text{as } z \to 0, \tag{35a}$$

$$M(z) \to M_1 \text{ (algebraically) as } z \to 1; \tag{35b}$$

$$N(\zeta) = N_0(\zeta) \to C\zeta^2 + \ldots \to 0 \quad \text{as } \zeta \to 0, \tag{36a}$$

$$N(\zeta) = N_0(\zeta) \to 1 \text{ (exponentially)} \quad \text{as } \zeta \to \infty. \tag{36b}$$

Here, $\kappa, \kappa_2, \ldots, M_1, C, \ldots$ are constants of order unity.

For the outer layer, based on (4), (31) can be written as

$$t = -\frac{\varepsilon}{R_\tau}\frac{df}{dz}, \tag{37a}$$

where, from (35) and (36),

$$\frac{\varepsilon}{R_\tau} = \frac{\varepsilon(z; R_\tau)}{R_\tau} \approx \kappa z\, M_0(z). \tag{37b}$$

In turn, the combination of (37) with (27) and (28) produces

$$t_0 = (1-z) = -\kappa z\, M_0(z)\frac{df_0}{dz}, \tag{38a}$$

$$t_1 = \frac{df_0}{dz} = -\kappa z\, M_0(z)\frac{df_1}{dz}, \ldots, \tag{38b}$$

such that the boundary-value problems for $f_0(z), f_1(z), \ldots$ are

$$\frac{df_0}{dz} = -\frac{(1-z)}{[\kappa z M_0(z)]}, \; f_0 \to 0 \quad \text{as } z \to 1; \tag{39a}$$

$$\frac{df_1}{dz} = \frac{(1-z)}{[\kappa z\, M_0(z)]^2}, \; f_1 \to 0 \quad \text{as } z \to 1; \ldots. \tag{39b}$$

The integration of (39) leads to

$$f_0(z) = \frac{1}{\kappa}[f_{00}(z) - f_{01}(z)]:$$

$$f_{00}(z) = \frac{[z - \log z]}{M_0(z)}, \; f_{01}(z) = (\kappa/M_1) + \int_z^1 \frac{M_0'(k)}{M_0(k)} f_{00}(k)\, dk: \tag{40a}$$

$$f_1(z) = -\frac{1}{\kappa^2}[f_{10}(z) - f_{11}(z)]:$$

$$f_{10}(z) = \frac{[(1/z) + \log z]}{M_0^2(z)}, \; f_{11}(z) = \left(\frac{\kappa}{M_1}\right)^2 + 2\int_z^1 \frac{M_0'(k)}{M_0(k)} f_{10}(k)\, dk; \ldots. \tag{40b}$$

Thus, for the eddy-viscosity closure model, the constants of (14a) are $A_{01} = 1/K_0 = 1/\kappa$, $A_0 = -f_{01}(0)/\kappa$, $W_0 = 1/\kappa, \ldots$, $J_1 = -1/\kappa^2$, $A_{11} = 1/K_1 = 1/\kappa^2$, $A_1 = f_{11}(0)/\kappa^2, \ldots$.

Correspondingly, for the inner layer,

$$s = \varepsilon \frac{\mathrm{d}g}{\mathrm{d}\zeta}, \tag{41a}$$

where

$$\varepsilon = \varepsilon(\zeta; R_\tau) \cong \kappa\zeta \left(1 + \frac{1}{R_\tau^2} \kappa_2 \zeta^2 + \ldots\right) N_0(\zeta). \tag{41b}$$

Thus, the boundary-value problems for $g_0(\zeta), g_1(\zeta), \ldots$ are

$$\frac{\mathrm{d}g_0}{\mathrm{d}\zeta} = \frac{1}{[1 + \kappa\zeta N_0(\zeta)]}, \; g_0 \to 0 \text{ as } \zeta \to 0; \tag{42a}$$

$$\frac{\mathrm{d}g_1}{\mathrm{d}\zeta} = -\frac{\zeta}{[1 + \kappa\zeta N_0(\zeta)]}, \; g_1 \to 0 \text{ as } \zeta \to 0; \ldots. \tag{42b}$$

The solutions of these equations are

$$g_0(\zeta) = \frac{1}{\kappa}[g_{00}(\zeta) + g_{01}(\zeta)]:$$

$$g_{00}(\zeta) = \log(1 + \kappa\zeta), \; g_{01}(\zeta) = \kappa^2 \int_0^\zeta \frac{v[1 - N_0(v)]\,\mathrm{d}v}{(1 + \kappa v)[1 + \kappa v N_0(v)]}; \tag{43a}$$

$$g_1(\zeta) = -\frac{1}{\kappa^2}[g_{10}(\zeta) + g_{11}(\zeta)]:$$

$$g_{10}(\zeta) = [\kappa\zeta - \log(1 + \kappa\zeta)], \; g_{11}(\zeta) = \kappa^3 \int_0^\zeta \frac{v^2(1 - N_0(v)]\,\mathrm{d}v}{(1 + \kappa v)[1 + \kappa v N_0(v)]}; \ldots. \tag{43b}$$

From (43), it is seen that the constants of (14b) are $B_{01} = 1/K_0 = 1/\kappa$, $B_0 = [g_{01}(\infty) + \log\kappa]/\kappa$, $V_0 = 1/\kappa^2, \ldots$, $H_1 = -1/\kappa$, $B_{11} = 1/K_1 = 1/\kappa^2$, $B_1 = -[g_{11}(\infty) - \log\kappa]/\kappa^2, \ldots$.

For the intermediate layer, based on (22), (31) can be written as

$$\left(1 - \frac{1}{R_\tau^{1/2}} T\right) = \frac{\varepsilon}{R_\tau^{1/2}} \frac{\mathrm{d}U}{\mathrm{d}\eta}, \tag{44a}$$

where, from (35) and (36),

$$\frac{\varepsilon}{R_\tau^{1/2}} = \frac{\varepsilon(\eta; R_\tau)}{R_\tau^{1/2}} \cong \kappa\eta \left(1 + \frac{1}{R_\tau} \kappa_2 \eta^2 + \ldots\right). \tag{44b}$$

The combination of (44) with (24) and (25) produces

$$dU_0/d\eta = \frac{1}{(\kappa\eta)}, \tag{45a}$$

$$dU_1/d\eta = -1 + \kappa\eta^2/(\kappa\eta)^2, \tag{45b}$$

$$dU_2/d\eta = \frac{1+\kappa\eta^2}{(\kappa\eta)^3} - \frac{\kappa_2\eta^2}{(\kappa\eta)}, \ldots. \tag{45c}$$

Integration of (45) leads to the following solutions for the velocity functions:

$$U_0 = \frac{1}{\kappa}\log\eta + U_{00}, \tag{46a}$$

$$U_1 = -\left(\frac{\eta}{\kappa} - \frac{1}{\kappa^2\eta}\right) + U_{10}, \tag{46b}$$

$$U_2 = \frac{1}{\kappa^2}\log\eta + U_{20} - \frac{1}{2}\left(\frac{\kappa_2\eta^2}{\kappa} + \frac{1}{\kappa^3\eta^2}\right), \ldots. \tag{46c}$$

Here, $U_{00}, U_{10}, U_{20}, \ldots$ are constants of integration. A comparison of (26) and (46) indicates that, for the eddy-viscosity closure model adopted here,

$$A_{01} = B_{01} = 1/K_0 = 1/\kappa, \quad \frac{1}{2}(B_0 - A_0) = U_{00} = \frac{1}{2\kappa}[g_{01}(\infty) + f_{01}(0) + \log\kappa];$$

$$W_0 = -H_1 = 1/\kappa, \; J_1 = -V_0 = -1/\kappa^2, \; U_{10} = 0;$$

$$A_{11} = B_{11} = 1/K_1 = 1/\kappa^2, \; \frac{1}{2}(B_1 - A_1) = U_{20} = -\frac{1}{2\kappa^2}[g_{11}(\infty) + f_{11}(0) - \log\kappa],$$

$$C_0 = E_2 = \kappa_2/\kappa, \; L_2 = -D_0 = 1/\kappa^3; \ldots. \tag{47}$$

From (25) and (45), the Reynolds stress functions are

$$T_0 = 1 + \kappa\eta^2/(\kappa\eta), \tag{48a}$$

$$T_1 = -1 + \kappa\eta^2/(\kappa\eta)^2, \tag{48b}$$

$$T_2 = \frac{1+\kappa\eta^2}{(\kappa\eta)^3} - \frac{\kappa_2\eta^2}{(\kappa\eta)}, \ldots. \tag{48c}$$

Thus, for the eddy-viscosity closure model adopted,

$$S(\eta; R_\tau) = \frac{\tau(\eta; R_\tau)}{u_\tau^3(R_\tau)} = 1 - \frac{1}{R_\tau^{1/2}}\frac{1+\kappa\eta^2}{(\kappa\eta)} + \frac{1}{R_\tau}\frac{1+\kappa\eta^2}{(\kappa\eta)^2} + O\left(\frac{1}{R_\tau^{3/2}}\right). \tag{49}$$

In turn, the first derivative of S with respect to η is

$$\frac{dS}{d\eta} = \frac{1}{R_\tau^{1/2}} \frac{1-\kappa\eta^2}{\kappa\eta^2} - \frac{1}{R_\tau} \frac{2}{\kappa^2\eta^3} + O\left(\frac{1}{R_\tau^{3/2}}\right). \tag{50}$$

From (49) and (50), it is seen that $\eta \to \eta_{\max}(R_\tau)$, $S(\eta; R_\tau) \to S_{\max}(R_\tau)$, a maximum, with

$$\eta_{\max}(R_\tau) = \frac{1}{\kappa^{1/2}}\left[1 - \frac{1}{\kappa^{1/2} R_\tau^{1/2}} + O\left(\frac{1}{R_\tau}\right)\right], \tag{51a}$$

$$S_{\max}(R_\tau) = 1 - \frac{2}{\kappa^{1/2} R_\tau^{1/2}} + \frac{2}{\kappa R_\tau} + O\left(\frac{1}{R_\tau^{3/2}}\right). \tag{51b}$$

For the purpose of comparison, it is determined that the employment of the mixing-length-theory eddy-viscosity closure model of Bush and Fendell [5] yields the following Reynolds stress in the intermediate layer:

$$S(\eta; R_\tau) = 1 - \frac{1}{R_\tau^{1/2}} \frac{1+\kappa\eta^2}{(\kappa\eta)} + \frac{1}{R_\tau} \frac{1+\kappa\eta^2}{2(\kappa\eta)^2} + O\left(\frac{1}{R_\tau^{3/2}}\right). \tag{52}$$

In turn, it is found that

$$\eta_{\max}(R_\tau) = \frac{1}{\kappa^{1/2}}\left[1 - \frac{1}{2\kappa^{1/2} R_\tau^{1/2}} + O\left(\frac{1}{R_\tau}\right)\right], \tag{53a}$$

$$S_{\max}(R_\tau) = 1 - \frac{2}{\kappa^{1/2} R_\tau^{1/2}} + \frac{1}{\kappa R_\tau} + O\left(\frac{1}{R_\tau^{3/2}}\right). \tag{53b}$$

From the preceding, it is seen that the difference in the Reynolds stress of the zero-equation and mixing-length-theory eddy-viscosity closure models is of $O(1/R_\tau)$; the difference in the location of the maximum Reynolds stress of the two models is of $O(1/R_\tau^{1/2})$.

4. Results and discussion

The asymptotic analysis presented shows that turbulent channel flow consists of three layers: an outer layer, an inner layer, and an intermediate layer. The outer and inner layers correspond to the outer and inner limits of classical theory and have the length scales h^* and ν^*/u_τ^*, respectively. The intermediate layer corresponds to the distinguished intermediate limit of the classical theory and has a length scale of $(h^*\nu^*/u_\tau^*)^{1/2}$. (It is seen that the intermediate-layer length scale is the geometric mean of the outer-layer and inner-layer length scales.) This distinguished intermediate layer is identified as being the one in which the turbulent and laminar contributions to the difference of the Reynolds stress from the wall stress are of the same order of magnitude.

The present analysis quantitatively yields the magnitude and location of the maximum of the Reynolds stress within the intermediate layer, in the process of determining its asymptotic behaviour in this layer. Further, through this three-layer analysis, it is found that the mean velocity achieves a value that is one-half of its

centreline value within this intermediate layer, in the process of determining its asymptotic behaviour in the layer.

The intermediate-layer velocity distribution, i.e., $(u-\frac{1}{2})/u_\tau$ as a function of η and R_τ, obtained from the experimental data of Laufer [6] is displayed in figure 1. Also displayed in this figure is the velocity distribution in this layer as determined from §2 of the present analysis,

$$\frac{(u-\frac{1}{2})}{u_\tau} = \left[\frac{1}{K_0}\log\eta + \frac{1}{2}(B_0 - A_0)\right] + O\,(R_\tau^{-1/2}), \tag{54}$$

with $K_0 = 0.41$ and $B_0 = 5.0$, $A_0 = 0.65$, such that $1/K_0 = 2.4$, $\frac{1}{2}(B_0 - A_0) = 2.2$. The figure shows that a substantial logarithmic region with universal slope and intercept does, in fact, exist.

The determination of the friction velocity $u_\tau(R)$ and the turbulent Reynolds number $R_\tau(R)$ is independent of whether a three-layer or a two-layer picture of turbulent channel flow is considered. It is found that

$$u_\tau \cong \frac{K_0}{\log R}\left[1 + \frac{\log\log R - \{K_0(A_0+B_0)+\log K_0\}}{\log R} + \ldots\right] \to 0, \tag{54a}$$

$$R_\tau \cong \frac{K_0 R}{\log R}\left[1 + \frac{\log\log R - \{K_0(A_0+B_0)+\log K_0\}}{\log R} + \ldots\right] \to \infty \tag{54b}$$

in the limit R, $\log R \to \infty$, which establishes the validity of the asymptotic analyses presented.

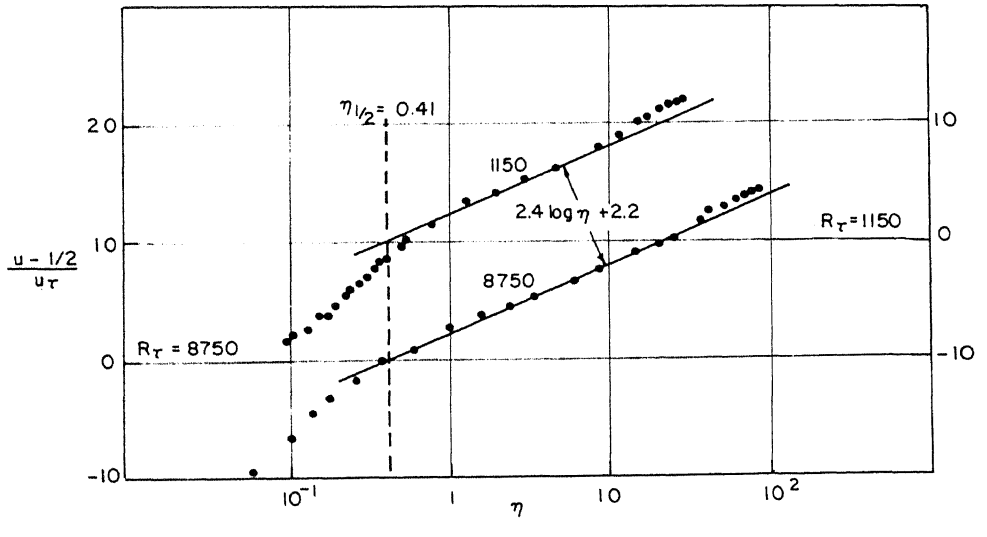

Figure 1. Intermediate-layer distribution as a function of η and R_τ (Laufer [6]).

References

[1] Afzal N, Millikan's argument at moderately large Reynolds number, *Phys. Fluids* **19** (1976) 600–602
[2] Afzal N, Fully developed turbulent flow in a pipe: an intermediate layer, *Ingen. -Archiv.*, **52** (1982) 355–377
[3] Afzal N, A sub-boundary layer within a two-dimensional turbulent boundary layer: an intermediate layer, *J. Mech. Theor. Appl.* **1** (1982) 963–973
[4] Bush W B and Fendell F E, Asymptotic analysis of turbulent channel and boundary-layer flow, *J. Fluid Mech.* **56** (1972) 657–681
[5] Bush W B and Fendell F E, Asymptotic analysis of turbulent channel flow for mixing length theory, *SIAM J. Appl. Math.* **26** (1974) 413–427
[6] Laufer J, The structure of turbulence in fully developed pipe flow, NACA Report, 1174 (1954)
[7] Long R R and Chen T C, Experimental evidence for the existence of the "mesolayer" in turbulent systems, *J. Fluid Mech.* **105** (1981) 19–59

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